The Inequalities of Positive Semi-Definite Block Matrix with Partial Order Relations

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Abstract: In 2019, Zübeide Ulukök obtained an important theorem in reference [1]: When \( H = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \) is a positive semi-definite matrix, then \( H' \leq \lambda(A) + \lambda(B) \begin{bmatrix} A \\ B \end{bmatrix} \), where \( A, B \) are n-order square matrices. In this paper, we firstly do the same thing for a \( 3 \times 3 \) positive semi-definite block matrix, and give a generalization of the above theorem. Next, we further generalize the case of \( k \times k \) positive semi-definite block matrix, and discuss the partial ordering relationship between the sum of matrices on quasi-diagonal lines and block matrices at other locations. Thus, we gave a new eigenvalue inequality.

Keywords: Positive Semi-Definite Matrix Hermitian Matrix Partial Order Relation

1. INTRODUCTION AND PRELIMINARIES

Inequalities of positive semi-definite block matrices have been widely used in matrix theory. In recent years, inequalities about block matrices have become a hot topic of research. At the same time, some very good results have been obtained, such as references [1, 3, 4, 5]. In this paper, we mainly discussed some positive semi-definite block matrices and obtained some matrix inequalities.

As we all know, the positive semi-definite block matrices have very good properties. Their eigenvalues are real numbers, so they can always be arranged in ascending order and we recorded them as \( \lambda_1(A) \leq \cdots \leq \lambda_n(A) \). In this paper, we use symbol \( \lambda_n(A) \) to represent the largest eigenvalue of a positive semi-definite matrix \( A \). And use symbol \( A \preceq B \) to represent \( B - A \) be a positive semi-definite matrix, obviously, \( "\preceq" \) is a partial order relation. In particular, \( A \succeq 0 \) denotes that matrix \( A \) is positive semi-definite. In addition, we call \( U \) a unitary matrix if it satisfies \( U^* U = I \), we call \( A \) a Hermitian matrix if it satisfies \( A^* = A \). Last, the \( A \oplus B \) denotes the direct sum of \( A \) and \( B \), the block diagonal matrix \( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \); and \( 0 \) represents a zero block matrix.

2. MAIN RESULTS

Let’s start with the following lemmas

**Lemma 1.1** Let \( A \in M_{m \times n} \) with \( m \geq n \), then \( \lambda(AA^*) = \lambda(A^*A) \oplus 0 \) with \( 0 \in M_{m-n} \).

**Lemma 1.2** Let \( A \in M_m \) be positive semi-definite matrix. Then \( \lambda_n(A)I \preceq A \preceq \lambda_1(A)I \), where \( I \) denotes...
The Inequalities of Positive Semi-Definite Block Matrix with Partial Order Relations

Lemma 1.3 If \( \begin{pmatrix} A & B \\ B & A \end{pmatrix} \) is a positive semi-definite matrix, then \( A \geq B \).

Proof Just consider the equation \( A - B = \frac{1}{2} (I - I) \begin{pmatrix} A & B \\ B & A \end{pmatrix} (I - I) \geq 0 \) and we can get the conclusion.

Lemma 1.4 If \( \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \) is a positive semi-definite matrix, \( A, C \) are square matrices of the same order, then we have \( A + C \geq B + B^* \) and \( A + C \geq -(B + B^*) \).

Proof Because \( \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \) is a positive semi-definite matrix, So, for any unitary matrix \( U \), we have

\[
U^* \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} U \geq 0 \, , \text{ thus there is } \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} + U^* \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} U \geq 0 \, . \text{ In particular, take}
\]

\[
U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
\]

then we have \( \begin{pmatrix} A + C & B + B^* \\ B + B^* & A + C \end{pmatrix} \geq 0 \). From lemma 1.3 we can get that

\[
A + C \geq B + B^* \, . \text{ And we can draw another conclusion by replacing the original } B \text{ with } -B.
\]

Lemma 1.5 If \( H \in M_n \) is a positive semi-definite matrix, then for any \( n \times k \) unitary matrix \( U \) satisfying \( U^* U = I_k \) and for each \( i = 1, \ldots, k \), we have \( \lambda_{i+m-k}(H) \leq \lambda_i(U^* HU) \leq \lambda_i(H) \).

Theorem 1 \( H = \begin{bmatrix} A & D \\ D^* & E \end{bmatrix} \) is a positive semi-definite matrix, where \( A, B, C \) are all \( n \)-order square matrices, then

\[
H' \leq 3[\lambda_i(A) + \lambda_i(B) + \lambda_i(C)]^{-1} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \text{ for } r \geq 1.
\]

Proof Because \( H \) is a positive semi-definite matrix, so there exists an invertible matrix \( P \) such that \( H = P^* P \). We divide \( P \) into blocks: \( P = [X, Y, Z] \), where \( X, Y, Z \in M_{n \times n} \), and we can know that

\[
X^* X = A, \ Y^* Y = B, \ Z^* Z = C. \text{ From Lemma 1.1, we can get the following results:}
\]

\[
\lambda_i(XX^*) = \lambda_i(X^* X \oplus 0) = \lambda_i(A), \ \lambda_i(YY^*) = \lambda_i(Y^* Y \oplus 0) = \lambda_i(B), \ \lambda_i(ZZ^*) = \lambda_i(Z^* Z \oplus 0) = \lambda_i(C).
\]

Noting that the following equation holds: \( H' = (P^* P)' = P^* (P P^*)^{-1} P \), and we denote
The Inequalities of Positive Semi-Definite Block Matrix with Partial Order Relations

\[ T = (PP^T)^{-1} \]. So, for \( H^r \), there is the following decompositions :

\[
H^r = P^r TP = \begin{bmatrix}
X^T X & X^T Y & X^T Z \\
Y^T X & Y^T Y & Y^T Z \\
Z^T X & Z^T Y & Z^T Z
\end{bmatrix} = \begin{bmatrix}
X & Y & Z \\
T & T & T \\
T & T & T
\end{bmatrix} \begin{bmatrix}
X \\
T \\
T
\end{bmatrix}
\]

We notice that

\[
\begin{bmatrix}
T & T & T \\
T & T & T \\
T & T & T
\end{bmatrix} = \begin{bmatrix}
T^1 & 0 & 0 \\
T^1 & 0 & 0 \\
T^1 & 0 & 0
\end{bmatrix}, \quad \text{and} \quad \lambda \begin{bmatrix}
T & T & T \\
T & T & T \\
T & T & T
\end{bmatrix} = 3\lambda(T)^r.
\]

So, for \( r \), there is the following decompositions :

\[
\begin{bmatrix}
X & Y & Z \\
T & T & T \\
T & T & T
\end{bmatrix} = \begin{bmatrix}
X \\
T \\
T
\end{bmatrix} = 3\lambda_r(T)
\]

So, from lemma 1.2. We can get that:

\[
H^r \leq 3\lambda_r(T) \begin{bmatrix}
A & B \\
C & C
\end{bmatrix} = 3\lambda_r(PP^T)^{-1} \begin{bmatrix}
A & B \\
C & C
\end{bmatrix}
\]

On the basis of this conclusion, we can easily get the results of Zübeyde Ulukök in reference[1]:

**Corollary 1** \( H = \begin{bmatrix}
A & C \\
C^* & B
\end{bmatrix} \) is a positive semi-definite matrix, where \( A, B \) are n-order square matrices, then

\[
H^r \leq 3[\lambda_r(A) + \lambda_r(B)]^{r-1} \begin{bmatrix}
A \\
B
\end{bmatrix} \quad \text{for} \quad r \geq 1.
\]

Next, we will discuss the case of \( k \times k \) positive semi-definite block matrices and explore the relationship between quasi-diagonal matrices and other block matrices at other locations.

**Theorem 2** \( H = \begin{bmatrix}
M_{11} & M_{12} & \cdots & M_{1k} \\
M_{21} & M_{22} & \cdots & M_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
M_{k1} & M_{k2} & \cdots & M_{kk}
\end{bmatrix} \) is a positive semi-definite matrix, and \( M_{ij}, \ldots, M_{kk} \) are all n-order square matrices, if it satisfied \( M_{ij} \) are all Hermitian matrices, for \( 1 \leq i \leq j \leq n \), then

\[
\sum_{i=1}^{k} M_{ii} \geq \frac{2}{k-1} \sum_{1 \leq i < j \leq k} M_{ij}
\]

**Proof** First of all, notice the following facts: if \( U_1, \ldots, U_k \) are all unitary matrices, \( A \) is a positive semi-definite matrix, then \( U^*_1AU_1 + \cdots U^*_kAU_k \) is still a positive semi-definite matrix.
After calculation, we can get that \( \sum_{i=1}^{k} \sum_{l=1}^{1} (M_{ii} + M_{il}) \geq \sum_{i=1}^{k} \sum_{l=1}^{1} (M_{il} + M_{il}) = 2 \sum_{1 \leq i < j \leq n} M_{ij} \).

So, there is \( \sum_{i=1}^{k} M_{ii} \geq \frac{2}{k-1} \sum_{1 \leq i < j \leq n} M_{ij} \).

**Corollary 2** If \( \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \) is a positive semi-definite matrix, \( A, C \) are square matrices of the same order, and \( B \) is a Hermitian matrix, then \( \text{tr}B \leq \max \{\text{tr}A,\text{tr}B\} \).

**Theorem 3** \( H = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{bmatrix} \) is a positive semi-definite matrix, and \( M_{11}, \ldots, M_{kk} \) are all \( n \)-order square matrices, if it satisfied \( M_{ij} \) are all Hermitian matrices, for \( 1 \leq i < j \leq n \), then for these \( 1 \leq i \leq n \), we have

\[
\frac{2}{k} \lambda_i (\sum_{1 \leq i < j \leq n} M_{ij}) \leq \lambda_i (H)
\]

**Proof** Take a unitary matrix \( U = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} \), then \( U^*U = I_n \), noticed that \( M_{ij} \) are all Hermitian matrices, for \( (1 \leq i < j \leq n) \), so we have

\[
U^*HU = \frac{1}{k} \sum_{i=1}^{k} M_{ii} + \frac{2}{k} \sum_{1 \leq i < j \leq n} M_{ij} \geq \frac{1}{k} \frac{2}{k-1} \sum_{1 \leq i < j \leq n} M_{ij} \sum_{1 \leq i < j \leq n} M_{ij} \quad \text{(from Theorem 2)}
\]

\[
= \frac{2}{k} \sum_{1 \leq i < j \leq n} M_{ij} \left( \frac{1}{k-1} + 1 \right) \geq \frac{2}{k} \sum_{1 \leq i < j \leq n} M_{ij} .
\]

Then for \( 1 \leq i \leq n \), from lemma 1.5 we have \( \frac{2}{k} \lambda_i (\sum_{1 \leq i < j \leq n} M_{ij}) \leq \lambda_i (U^*HU) \leq \lambda_i (H) \), thus the conclusion \( \frac{2}{k} \lambda_i (\sum_{1 \leq i < j \leq n} M_{ij}) \leq \lambda_i (H) \) is proved.

**REFERENCES**


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