Note on the Vector Sub-Space $K(H)^2$ of Compact Operators on a H Hilbert’s Infinite Dimension Separable Space

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Abstract: can we provide a vector space with two Hilbert’s space structures? Obviously, the vector space $K(H)$ of compact operators is candidate. The targeted vector spaces are noted $K(H)^1$ and $K(H)^2$. The following lines present the $(H)^2$ space.

Keywords: vector space, scalar products, norm, full norm, hilbertian basis, Cauchy-Schwarz inequality, compact operators, Banach’s space, Hilbert’s space.

1. PRELIMINERIES

1.1. Information

The vector space $B(H)$ of operators limited on Hilbert’s $H$ infinite dimension separable space is Hilbert’s space of which the scalar product and the full hermitian norm are respectively noted in this way:

$$<A, B> = \sum_{i=1}^{\infty} \frac{1}{2^i} (A(e_i), B(e_i))$$

for all $A, B \in B(H)$, $e_i \in b$ and

$$\|A\|_1 = \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \|A e_i\|^2\right)^{1/2}$$

for each $A \in B(H)$, $e_i \in b$, $b$ being a hilbertian basis. Sometimes, we will note these results respectively by $<,>_1$ and $\| \|_1$.

[Masamba Sala Voka Joseph]

2. CANDIDATE $K(H)$ VECTOR SPACE

As the question asked alludes to the existence of two compact vector spaces, we have found it appropriate to call and note them respectively $K(H)^1$ and $K(H)^2$. These notations do not add nor lessen the properties of a compact space $K(H)$; they inherit from all the properties due to a compact space noted $K(H)$. The following lines speak about the vector space $K(H)^2$.

3. RECALLS

3.1. Definition

Consider $A$ an operation attached to a separable Hilbert’s space $H$. It is said that $A$ is a compact operator if it changes any limited part $D$ of part $H$ into a pre-compact part $A(D)$, whereas $A$ is a finite rank dimension operator if its image $R(A)$ is finite.

3.2. Proposition

Consider $H$ a separable Hilbert’s space and $(A_n)$ a succession of compact operators which converge towards a linear operator $A$, then a compact operator.

3.3. Proposition

Consider $H$ a separable Hilbert’s space and $A$ a compact operator; then there exists a succession $(A_n)$ of finite rank operators such as $\lim_{n \to \infty} A_n = A$ [Dieudonné]
4. NOTE

As $K(H)^2$ is a closed vector sub–space of $B(H)$, it benefits from properties of Hilbert’s space $B(H)$ such as:

- The closed unity bowl is noted $B_H = x \in H : \|x\| \leq 1$
- The following proposition : for a compact operator $A$ and a vector $x \in B_H$ we have the inequality: $\|Ax\|^2 \leq (\sum_{i=1}^{\infty} |a_i| \|Ae_i\|)^2$
- The field of definition of a compact operator $A$ is dense in $H$ and for all the two compact operators $A, B \in K(H)^2$ and $e \in b$, the scalar $(A e_i, B e_i)$ is an element of $\mathbb{C}$.

5. REMARK

$K(H)^2$ is a vector sub–space of $B(H)$ so that the field of the definition of compact operators including those that are the terms of a succession $(A_n)$ of finite rank operators, are dense in $H$.

Thus, whatever an operator $A$ and a term $A_n$ of a succession $(A_n)$ of finite rank operators such as $\lim_{n \to \infty} A_n = A$ for any $n \in \mathbb{N}$, $D(A) = H$ and $D(A_n) = H$.

Now consider $A$ and $B$ two compact operators such as $\lim_{n \to \infty} A_n = A$, $\lim_{n \to \infty} B_n = B$ and a vector $e_i \in b$.

It is clear that $A_n e_i$ and $B_n e_i$ are vectors of $H$ since $A_n$ and $B_n$ are operators on $H$. It follows that $(A_n e_i, B_n e_i)$ is a scalar.

For this particular case of compact operators, we will use in the serial, the terms of two successions of finite rank operators $(A_n)$ and $(B_n)$ such as $\lim_{n \to \infty} A_n = A$ and $\lim_{n \to \infty} B_n = B$ whatever two compact operators $A$ and $B$.

Concretely, if for $A$ and $B$ we take respectively $A_n$ and $B_n$, then the number $< A, B >_1 = \sum_{i=1}^{\infty} \frac{1}{2^i} (A e_i, B e_i)$ takes the following form : $< A, B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (A_n e_i, B_n e_i)$ for all the two operators $A, B$ on $H$ and $e_i \in b$, $b$ being a Hilbertian basis. It is wise to be sure that this formula does not depend on the choice of finite rank operators $(A_n)$ and $(B_n)$ such as $\lim_{n \to \infty} A_n = A$ and $\lim_{n \to \infty} B_n = B$.

In fact, if $(P_n)$ and $(Q_n)$ are two other successions of finite rank operators such as $\lim_{n \to \infty} P_n = A$ and $\lim_{n \to \infty} Q_n = Q$, it is clear that if $\lim_{n \to \infty} P_n = A$ and $\lim_{n \to \infty} Q_n = B$, then it must absolutely get the following equalities:

$A = P$ and $B = Q$. These equalities are true since the limit of a succession is unique; they involve the equalitie $< A, B > = < P, Q >$ that clearly means that: $\sum_{i=1}^{\infty} \frac{1}{2^i} (A_n e_i, B_n e_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} (P_n e_i, Q_n e_i)$.

We have found it appropriate to name this scalar product in this way : $< A, B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (A_n e_i, B_n e_i)$ for all two compact operators $A, B$ and $e_i \in b$; sometimes, we note it simply $< , >_2$.

6. STRUCTURES CONFERRED TO A COMPACT VECTOR SPACE $K(H)^2$

6.1. Theorem

Consider $K(H)^2$ Hilbert’s infinite dimension separable complex spaces, $A$ and $B$ two compact operators such as $A = \lim_{n \to \infty} A_n$, $B = \lim_{n \to \infty} B_n$ and $e_i \in b$; and the following application : $<, >_2 : (K(H)^2)^2 \to \mathbb{C}$ such as $< A, B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (A e_i, B e_i)$ for $A, B \in K(H)^2$ and $e_i \in b$; then $<, >_2$ is a scalar product on $K(H)^2$.

PROOF

(i) Whatever three compact operators $A, B, C \in K(H)$ and $e_i \in b$ :

$(i_1) < A + B, C >_2 = \sum_{n=1}^{\infty} \frac{1}{2^n} (A_n e_i, C_n e_i) + \sum_{n=1}^{\infty} \frac{1}{2^n} (B e_i, C_n e_i)$
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\[< A, C >_2 + < B, C >_2 \]

\((i_2)\) \( < A, B + C >_2 = \)

\[= \sum_{n=1}^{\infty} \frac{1}{2^n} (A_n e_i, B_n e_i) + \sum_{n=1}^{\infty} \frac{1}{2^n} (A_n e_i, C_n e_l) \]

\[= < A, B >_2 + < A, C >_2 \]

\((ii)\) Whatever two compact operators \(A, B\), a scalar \(t\) and \(e_i \in b\):

\((i_{ii})\) \(< tA, B >_2 = \)

\[= t \sum_{n=1}^{\infty} \frac{1}{2^n} (A_n e_i, B_n e_i) = t < A, B >_2 \]

\((ii_2)\) \(< A, tB >_2 = \)

\[= \sum_{n=1}^{\infty} \frac{1}{2^n} (A_n e_i, tB_n e_i) = t^2 < A, B >_2 \]

\((iii)\) Whatever two compact operators \(A, B\) and \(e_i \in b\):

\(< A, B >_2 = \)

\[= \sum_{n=1}^{\infty} \frac{1}{2^n} (A_n e_i, B_n e_i) = \sum_{n=1}^{\infty} \frac{1}{2^n} (B_n e_i, A_n e_i) \]

\((iv)\) Whatever a compact operator \(A\) and \(e_i \in b\):

\(< A, A >_2 = \sum_{n=1}^{\infty} \frac{1}{2^n} \| A_n e_i \|^2 > 0 \); it follows that \(\| A_n e_i \|^2 > 0\) for any \(e_i \in b\). The equality \(< A, A >_2 = 0\) means that we have \(< A, A >_2 = \sum_{n=1}^{\infty} \frac{1}{2^n} (A_n e_i, B_n e_i) = \sum_{n=1}^{\infty} \frac{1}{2^n} \| A_n e_i \|^2 = 0\) whatever \(e_i \in b\) or simply \(A = 0_H\).

The norm associated with the scalar productions is therefore noted \(\| A \|_2 = (\sum_{n=1}^{\infty} \frac{1}{2^n} \| A_n e_i \|^2)^{1/2}\) for any compact operator \(A\) and \(e_i \in b\).

7. REMARK

Consider the two norms \(\| \cdot \|_2\) and \(\| \cdot \|_1\) on \(K(H)^2\); in this work, the word ‘norm’ means a structure of general topology or functional analysis on the one hand, on the other hand in arithmetic’s, a real number on the other hand: for instance: \(\| \cdot \|_1 \leq \| \cdot \|_2\) and \(\| \cdot \|_2 \leq \| \cdot \|_1\) which means that: \(\| \cdot \|_1 = \| \cdot \|_2\).

These elements, the proposition (2) of (4 Note) and the comparison of norms have helped enough us to get the desired results:

8. COMPARISON OF NORMS \(\| \cdot \|_1\) and \(\| \cdot \|_2\)

8.1. Theorem

Consider \(H\) a separable Hilbert’s space on \(k\) and \(\| \cdot \|_1\) and \(\| \cdot \|_2\) the two norms on the compact vector space \(K(H)^2\), then we have: \(\| \cdot \|_1 \leq \| \cdot \|_2\).

PROOF

In fact, for any compact operator \(A\) and a vector \(x\) of \(H\) belonging to \(B_H\), we have the following equalities and inequalities:

\[\| Ax \|^2 \leq (\sum_{i=1}^{\infty} |a_i| \| Ae_i \|)^2\]

[Porposition (2) the 4. note]

\[\leq (\sum_{i=1}^{\infty} |a_i| \| Ae_i \|)^2 = (\sum_{i=1}^{\infty} |a_i| \| A \|_2 \| e_i \|)^2\]

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\[ \text{[since } \|A\|_2 = \sup\{\|Ax\| : i = 1, 2, 3, \ldots \}) \]

\[ = \sum_{i=1}^{\infty} |a_i|^2 \|A\|_2^2 = \|A\|_2^2 \]

\[ \text{[since } \sum_{i=1}^{\infty} |a_i|^2 = 1 = \|e_i\|^2 ] \]

Brief:

\[ \|Ax\|^2 \leq \|A\|_2^2 \] or simply \( \|Ax\| \leq \|A\|_2 \); it follows that the norm

\[ \|A\|_1 = \sup\{\|Ax\| : \|x\| \leq 1\} \leq \|A\|_2 \]

or simply the result:

\[ \|A\|_1 \leq \|A\|_2. \]

**8.2. Theorem**

Consider \( H \) a separable Hilbert’s space on \( \mathbb{K} \) and \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) the two numbers on the vector space \( K(H)^2 \): then, we have:

\[ \| \cdot \|_2 \leq \| \cdot \|_1. \]

**Proof**

It is easy to note that if \( A \) is a compact operator and \( x \) a vector of \( H \) belonging to \( B_H \), we have the following inequalities and equalities:

\[ \|A\|_2^2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Ae_i) \]  

[by definition of \( \| \cdot \|_2 \)]

\[ \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|^2 \]

[Cauchy–Schwarz’s inequality]

\[ = \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|^2 \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|^2 \]

\[ = \|A\|_1 \|e_i\|^2 \sum_{i=1}^{\infty} \frac{1}{2^i} \]

[since \( \|A\|_1 = \sup\{\|Ae_i\| : e_i \in b\} \)]

\[ = \|A\|_1^2 \]

[since \( \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 = \|e_i\|^2 \)]

finally \( \|A\|_2^2 \leq \|A\|_1^2 \) or simply \( \|A\|_2 \leq \|A\|_1. \)

**9. Conclusion**

**9.1. Remark**

According as we consider a norm a numerical number, a topological structure, a functional analysis structure and, given results got, we present the conclusion in these words:

**9.2. Theorem**

The inequalities \( \| \|_1 \leq \| \|_2 \) and \( \| \|_2 \leq \| \|_1 \) mean that the two numbers \( \| \|_1 \) and \( \| \|_2 \) are equal or simply \( \| \|_1 = \| \|_2. \)

**9.3. Theorem**

The two norms \( \| \|_1 \) and \( \| \|_2 \) are hermitian and full; the vector space \( K(H)^2 \) provided with the norm \( \| \|_2 \) is a Banach’s space and, also the vector space \( K(H)^2 \) provided with the norm \( \| \|_2 \) is a Banach’s space.

**9.4. Theorem**

The two norms \( \| \|_1 \) and \( \| \|_2 \) are hermitian and full; the vector space \( K(H)^2 \) provided with the norm \( \| \|_2 \) is a Hilbert’s space and, also the vector space \( K(H)^2 \) provided with the norm \( \| \|_2 \) is a Hilbert’s space.
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