Power Mean Labeling of Identification Graphs

P.Mercy¹, S. Somasundaram²

¹,² Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, India

*Corresponding Author: P.Mercy, Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, India

Abstract: A graph \( G = (V, E) \) is called a Power mean graph with \( p \) vertices and \( q \) edges, if it is possible to label the vertices \( x \in V \) with distinct elements \( f(x) \) from \( 1, 2, 3, \ldots, q+1 \) in such a way that when each edge \( e = uv \) is labeled with

\[
 f(e = uv) = \left( \frac{1}{f(u)^q f(v)^q + f(v)} \right)^{q+1}
\]

or

\[
 f(e = uv) = \left( \frac{1}{f(u)^q f(v)^q + f(v)} \right)^{q+1}
\]

so that the resulting edge labels are distinct. Here \( f \) is called a Power mean labeling of \( G \). We investigate

Power mean labeling for some standard graphs.

Keywords: Power mean labeling, Power mean graph, union of graphs, union of \( m \) copies of cycles.

AMS subject classification (2010): 05C38, 05C76, 05C78

1. INTRODUCTION

The graphs considered here are finite and undirected graphs. Let \( G = (V, E) \) be a graph with \( p \) vertices and \( q \) edges. For a detailed survey of graph labeling we refer to Gallian [2] and Acharya et al.[1]. For all other standard terminology and notations we follow Harary [3]. In [6] Somasundaram and Ponraj introduced and studied [9] mean labeling for some standard graphs. Sandhya and Somasundaram [5] introduced Harmonic mean labeling of graphs and Sandhya et al. [4] studied the technique in detail. Somasundaram et al.[7] introduced the concept of Geometric mean labeling of graphs and studied their labeling in [8]. In this paper we define Power mean labeling and investigate some standard graphs for \( C_n^2, C_m \oplus P_n, C_m \circ P_n \), and \( C_m \) and \( C_n \) sharing a common edge for power mean labeling. We provide illustrative examples to support our study.

2. DEFINITION AND RESULTS

Now we introduce the main concept of this paper

Definition 2.1. A graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is said to be a Power Mean Graph if it is possible to label the vertices \( x \in V \) with distinct labels \( f(x) \) from \( 1, 2, 3, \ldots, q+1 \) is such a way that when each edge \( e = uv \) is labeled with

\[
 f(e = uv) = \left( \frac{1}{f(u)^q f(v)^q + f(v)} \right)^{q+1}
\]

or

\[
 f(e = uv) = \left( \frac{1}{f(u)^q f(v)^q + f(v)} \right)^{q+1}
\]
Power Mean Labeling of Identification Graphs

so that the resulting edge labels are distinct. In this case, \( f \) is called Power mean labeling of \( G \).

**Remark 2.1.** If \( G \) is a Power mean labeling graph, then 1 must be a label of one of the vertices of \( G \), since an edge should get label 1.

**Remark 2.2.** If \( p > q + 1 \), then the graph \( G = (p, q) \) is not a Power mean graph, since it does not have sufficient labels from \( \{1, 2, 3, \ldots, q + 1\} \) for the vertices of \( G \).

The following results will be used in the edge labelings of some standard graphs to get Power mean labeling.

**Proposition 2.1.** Let \( a, b \) and \( i \) be the positive integers with \( a < b \). Then

\[
\begin{align*}
(i) & \quad a \leq (a^b b^a) < b, \\
(ii) & \quad i < (i^l + 2(i + 2)^l)^{2i + 1} < (i + 1), \\
(iii) & \quad i < (i^l + 3(i + 3)^l)^{2i + 1} < (i + 2), \\
(iv) & \quad i < (i^l + 4(i + 4)^l)^{2i + 1} < (i + 1), \text{ and} \\
(v) & \quad (i^l + i)^{\frac{1}{l}} + 1 < (i+1)^{\frac{1}{l}}.
\end{align*}
\]

**Proof.** (i) Since \( ab + a = ab < b a = b^a a^b = b^{a+b} \), we get the inequality in Proposition 2.1.(i). That is, the Power mean of two numbers lies between the numbers \( a \) and \( b \). Thus we infer that if vertices \( u, v \) have labels \( i, i + 1 \) respectively, then the edge \( uv \) may be labeled \( i \) or \( i + 1 \) for Power mean labeling.

(ii) As a proof of this inequality, we see

\[
\begin{align*}
i^l + 2(i + 2)^l & < i^l [i(i + 2)]^l, \\
& < i^l (i + 1)^{2i}, \text{ since } i(i + 2) < (i + 1)^2, \\
& < (i + 1)^2 (i + 1)^{2i}, \\
& = (i + 1)^{2i + 2}.
\end{align*}
\]

This leads to \( (i^l + 2(i + 2)^l)^{\frac{1}{l+2}} < (i + 1) \). Therefore, if \( u, v \) have labels \( i, i + 2 \) respectively, then the edge \( uv \) may be labeled \( i \) and \( i + 1 \).

(iii) Next we have

\[
\begin{align*}
i^l + 3(i + 3)^l & = i^l [i(i + 3)]^l, \\
& < i^l (i + 2)^{2i}, \text{ since } i(i + 3) < (i + 2)^2, \\
& < (i + 2)^3 (i + 2)^{2i}, \\
& = (i + 2)^{2i + 3}.
\end{align*}
\]

This leads to \( (i^l + 3(i + 3)^l)^{\frac{1}{2i+3}} < (i + 2) \). Hence, if \( u, v \) have labels \( i, i + 3 \) respectively, then the edge \( uv \) may be labeled \( i + 1 \) without ambiguity.

(iv) Now

\[
\begin{align*}
i^l + 4(i + 4)^l & = i^l [i(i + 4)]^l, \\
& < i^l (i + 2)^{2i}, \text{ since } i(i + 4) < (i + 2)^2, \\
& < (i + 2)^4 (i + 2)^{2i}, \\
& = (i + 2)^{2i + 4}.
\end{align*}
\]

Therefore
Power Mean Labeling of Identification Graphs

\[ |i^i + 4(i + 4)^i|^{2i + 4} < i + 2. \]

Hence if \( u, v \) have labels \( i, i + 4 \) respectively, then the edge \( uv \) may be labeled \( i + 1 \).

(v) Now

\[
2^{i+1} = (i + 1)^{i+1},
\]

\[
= 1 + (i+1)C_i + \cdots + (i+1)C_{i+1},
\]

\[
\geq 1 + i + \cdots + (i + 2) \text{ terms},
\]

\[
\geq i + 2 > i.
\]

Therefore \( (i^i i^i)^i + 1 = i i^i + 1 < 2. \) Thus we observe that if \( u, v \) are labeled \( 1, 1 \) respectively, then the edge \( uv \) may be labeled 1 or 2.

3. IDENTIFICATION OF TWO GRAPHS

In this section we study the power mean labeling of some identification graphs.

**Theorem 3.1.** Let \( G_1 = (p_1, q_1) \) and \( G_2 = (p_2, q_2) \) be any two graphs with power mean labeling \( f \) and \( g \) respectively. Let \( u \) and \( v \) be the vertices of \( G_1 \) and \( G_2 \) respectively, such that \( f(u) = g(v) = q \). Then the graph \( (G_1)f \ast (G_2)g \) obtained from \( G_1 \) and \( G_2 \) by identifying the vertices \( u \) and \( v \) is a power mean graph.

**Proof.** Obviously \((G_1)f \ast (G_2)g \) has \( p_1 + p_2 - 1 \) vertices and \( q_1 + q_2 \) edges. Let the vertex set of \( G_1 \) be \( V(G_1) = \{ u_i : 1 \leq i \leq p_1 - 1 \} \) and that of \( G_2 \) be \( V(G_2) = \{ v_i : 1 \leq i \leq p_2 - 1, \} \).

Define a function

\[ h : V((G_1)f \ast (G_2)g) \to \{ 1, 2, 3, \ldots, q_1 + q_2 - 1 \} \]

by \( h(u_i) = f(u_i), 1 \leq i \leq p_1 - 1, h(v_i) = q_1 + g(v_i), 1 \leq i \leq p_2 - 1. \) Then edge labels of \( G_1 \) are \( 1, 2, 3, \ldots, q_1 \) and edge labels of \( G_2 \) are \( q_1 + 1, q_2 + 2, q_3 + 3, \ldots, q_1 + q_2. \) Hence \((G_1)f \ast (G_2)g \) is a power mean graph.

3.1 Power mean labeling for \( C_n^2 \)

In this section, we prove the power mean labeling of common vertices between two cycles with \( n \) number of vertices and illustrate with examples.

**Theorem 3.2.** The graph \( C_n^{(2)} \) is a Power mean graph.

**Proof.** Let \( u \) be the central vertex of \( C_n^{(2)} \). Let the vertices of first cycle be \( u_1, u_2, u_3, \ldots, u_n \) and the vertices of second cycle be \( w_1, w_2, w_3, \ldots, w_n \). Each cycle is a Power mean graph. Let \( f \) be the corresponding Power mean labeling of the cycle. Take \( G_1 = G_2 = C_n \) then \((G_1)f \ast (G_2)f = C_n \) then \((G_1)f \ast (G_2)f = C_n^{(2)} \). By Theorem 3.1, \( C_n^{(2)} \) is a Power mean graph.

**Example 3.1.** A Power mean labeling of \( C_6^{(2)} \) is given in Figure 3.1.

![Figure 3.1: \( C_6^2 \)](image-url)
Power Mean Labeling of Identification Graphs

Example 3.2. A power mean labeling of $C_3^{(2)}$ is given in Figure 3.2

![Figure 3.2: $C_3^{2}$](image)

3.2 Power mean labeling for Two cycles $C_n$ and $C_m$ sharing a common edge

In this section, we prove the power mean labeling of common edge between two cycles with different number of vertices and provide an example.

**Theorem 3.3.** Two cycles $C_n$ and $C_m$ sharing a common edge admit Power mean labeling.

**Proof.** Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of cycle $C_n$. Let $w_1, w_2, w_3, \ldots, w_m$ be the vertices of cycle $C_m$. Let $G$ be the graph sharing a common edge of the two cycles. Without loss generality, assume that $e = v_{n-1}v_n$ is the common edge between $C_n$ and $C_m$. Define a function $f : V(G) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ as

(i) $f(v_i) = i$, $1 \leq i \leq n - 1$.
(ii) $f(v_n) = n + 1$.
(iii) $f(w_i) = n + i$, $2 \leq i \leq n - 1$, and
(iv) $f(v_{n-1}) = f(w_1)$ and $f(v_n) = f(w_1)$

By Proposition 2.1 (i) and (v), the edges are labeled.

(i) $E(v_iv_{i+1}) = i + 1$, $1 \leq i \leq n - 2$
(ii) $E(v_{n-1}v_n) = n$
(iii) $E(v_nv_1) = 1$
(iv) $E(w_{i-1}w_i) = n + i$, $2 \leq i \leq n$, and
(v) $E(w_{m-1}w_m) = n + 1$.

As the edges are distinct, the graph $G$ is a Power mean graph.

**Example 3.3.** A graph $G$ sharing a common edge between the cycles $C_5$ and $C_8$ is explained in Figure 3.3.

![Figure 3.3: $C_7$ and $C_8$](image)
3.3 Power mean labeling for Dragon \(C_m \circ P_n\)

**Dragon:** A dragon is a graph formed by joining an end vertex of a path \(P_n\) with a vertex of the cycle \(C_m\). It is denoted by \(C_m \circ P_n\).

Here, we prove the power mean labeling of dragon and provide an illustrative example.

**Theorem 3.4.** A Dragon \(C_m \circ P_n\) is a Power mean graph.

**Proof.** Let \(G = C_m \circ P_n\) be the given graph. Let \(u_1, u_2, u_3, \ldots, u_m\) be the vertices of cycle \(C_m\). Let \(w_1, w_2, w_3, \ldots, w_n\) be the vertices of path \(P_n\). Here \(u_m = w_1\). Define a function \(f: V(C_m \circ P_n) \rightarrow \{1, 2, 3, \ldots, q + 1 = 2(n + 1)\}\) as

(i) \(f(u_i) = i, \quad 1 \leq i \leq m\)

(ii) \(f(w_{i+1}) = m + i, \quad 1 \leq i \leq n - 1\)

(iii) \(f(u_m) = f(w_1)\).

We get the edge labels as

(i) \(E(u_i u_{i+1}) = i + 1; 1 \leq i \leq m - 1\)

(ii) \(E(w_i w_{i+1}) = m + i; 1 \leq i \leq n - 1\)

(iii) \(E(u_m u_1) = 1\)

By Proposition 2.1.(i) and (v), the edge labels are distinct. The Dragon \(C_m \circ P_n\) is a Power mean graph.

**Example 3.4.** The graph Dragon \(C_m \circ P_n\) is given in Figure 3.4.

![Dragon Graph](image)

**Figure 3.4: Dragon \(C_m \circ P_n\)**

3.4 Power mean labeling for \(C_m \circ P_n\)

In this section, we establish the power mean labeling of the graph \(G\). It is obtained by identifying a pendant vertex of \(P_n\) and a vertex of \(C_m\) and illustrate with an example.

**Theorem 3.5.** Let \(G\) be a graph obtained by identifying a pendant vertex of \(P_n\) and a vertex of \(C_m\). The graph \(G\) admits a Power mean graph.

**Proof.** Let \(u_1, u_2, u_3, \ldots, u_m\) be the vertices of \(C_m\) and \(v_1, v_2, v_3, \ldots, v_n\) be the vertices of \(P_n\). Here we may take \(u_m = v_1\).

Define a function \(f: V(G) \rightarrow \{1, 2, 3, \ldots, q + 1\}\) by

(i) \(f(u_i) = i; 1 \leq i \leq m\).

(ii) \(f(v_{i+1}) = m + i; 1 \leq i \leq n - 1\).

By Proposition 2.1.(i) and (v), the edges are labeled

(i) \(E(u_i u_{i+1}) = i + 1; 1 \leq i \leq m - 1\)

(ii) \(E(u_m u_1) = 1\)

(iii) \(E(v_1 v_{i+1}) = m + i; 1 \leq i \leq n - 1\)
Power Mean Labeling of Identification Graphs

As the edges are distinct, the graph $C_m \circ P_H$ is a Power mean graph.

4. CONCLUSION

In this paper we have proved that $C_m^{2}$, $C_m @ P_H$, $C_m \circ P_H$, and $C_m$ and $C_H$ sharing a common edge are amenable for Power mean labeling. Also illustrative examples are provided.

REFERENCES


AUTHORS’ BIOGRAPHY

Ms. P. Mercy is a Research Scholar in Mathematics at Manonmaniam Sundaranar University, Tirunelveli. She works in Graph theory, particularly at present, in Power Mean Labeling of Graphs with the guidance of Prof. S. Somasundaram.

Dr. S. Somasundaram is a Professor of Mathematics at Manonmaniam Sundaranar University, Tirunelveli. His research interests include Harmonic Analysis, Fixed point theory and Graph theory. He has widely published in national and international journals.


Copyright: © 2018 Authors. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.