Fuzzy ideal of Partially Ordered Near-Ring

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Abstract: In this paper we introduce the notion of fuzzy ideal of a partially ordered near-ring (PON), T-fuzzy ideal of PON, normal T-fuzzy ideal of PON and fuzzy magnified translation. Also we study the characterizations partially ordered near-rings.

Keywords: Fuzzy ideal of PON, T-fuzzy ideal of PON, normal T-fuzzy ideal of PON and f-invariant.

1. INTRODUCTION


2. PRELIMINARIES

For the sake of continuity we recall the following definitions.

Definition 1. A non-empty set N with two binary operations “+” and “·” is called a near-ring if

(i) \((N, +)\) is a group (not necessarily abelian)
(ii) \((N, \cdot)\) is a semi group
(iii) \(x \cdot (y + z) = x \cdot y + x \cdot z\) for all \(x, y, z \in N\).

We will use the word “near-ring” to mean “left near-ring”.

Definition 2[4]. A t-norm is a function \(T: [0, 1] \times [0, 1] \rightarrow [0, 1]\) that satisfies the following condition for all \(x, y, z \in [0, 1]\)

(i) \(T(x, 1) = x\)
(ii) \(T(x, y) = T(y, x)\) (commutativity)
(iii) \(T(x, T(y, z)) = T(T(x, y), z)\) (associativity)
(iv) \(T(x, y) \leq T(x, z)\) whenever \(y \leq z\) (monotonicity).

Note that a t-norm \(T(0, 0) = 0\), \(T(1, 1) = 1\) and \(T(x, y) \leq \min(x, y)\).
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**Definition 3** [10]. Let $N$ be a near-ring. A near-ring $N$ is called a PON if

(i) $a \leq b$ then $a + g \leq b + g \quad \forall a, b, g \in N$

(ii) $a \leq b$ and $c \geq 0$ then $ac \leq bc$ and $ca \leq cb \quad \forall a, b, c \in N$.

**Definition 4** [8]. Let $\mu$ be a fuzzy subset of $X$ and $a \in \{0, 1 - \sup \{\mu(x)/x \in X\}\}$, $b \in [0, 1]$. The mappings

$\mu^a : X \to [0, 1]$, $\mu^b : X \to [0, 1]$ and $\mu^a \cdot b : X \to [0, 1]$ are called fuzzy translation, fuzzy multiplication and fuzzy magnified translation of $\mu$ respectively for all $x \in X$ respectively.

3. Fuzzy ideals of partially ordered near-rings

**Definition 5**. Let $N$ be a PON. A fuzzy subset $\mu$ of $N$ is said to be a fuzzy sub near-ring of $N$ if

(i) $\mu(x - y) \geq \min \{\mu(x), \mu(y)\}$

(ii) $\mu (x y) \geq \min \{\mu(x), \mu(y)\}$

(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ for all $x, y \in N$

**Definition 6**. Let $\mu$ be a non-empty fuzzy subset of a PON $N$. Then $\mu$ is called a fuzzy ideal of $N$ if

(i) $\mu(x - y) \geq \min \{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq \mu(y)$

(iii) $\mu(x + z)y - xy) \geq \mu(z)$

(iv) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ for all $x, y \in N$

Note that $\mu$ is fuzzy left ideal of $N$ if it satisfies (i), (ii) and (iv) and $\mu$ is fuzzy right ideal of $N$ if it satisfies (i), (iii) and (iv).

**Definition 7**. A fuzzy subset $\mu$ of PON $N$ is called T-fuzzy right (resp. left) ideal if

(i) $\mu(x - y) \geq T(\mu(x), \mu(y))$

(ii)$\mu ((x + z)y - x y) \geq \mu (z) (\mu(x y) \geq \mu(y))$

(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ for all $x, y, z \in N$.

If $\mu$ is a T-fuzzy left ideal and T-fuzzy right ideal of a PON then $\mu$ is called a T-fuzzy ideal of $N$.

**Theorem 1**: If $\{\mu_i : i \in I\}$ is a family of T-fuzzy ideal of PON $N$, then $V_{i \in I} \mu_i$ is also a T-fuzzy ideal of $N$ where $V_{i \in I} \mu_i$ is defined by $\left(\bigcup_{i \in I} \mu_i\right)(x) = \sup \{\mu_i(x) : i \in I\}$ for all $x \in N$.

**Proof.** Let $\{\mu_i : i \in I\}$ be a family of T-fuzzy ideal of a PON N. For any $x, y, z \in N$ then

(i) $\left(\bigcup_{i \in I} \mu_i\right)(x - y) = \sup \{\mu_i(x - y) : i \in I\}$

$\geq \sup \{T(\mu_i(x), \mu_i(y)) : i \in I\}$

$= T\{\sup \mu_i(x) : i \in I, \sup \mu_i(y) : i \in I\}$

$= T\left(\bigcup_{i \in I} \mu_i\right)(x), \left(\bigcup_{i \in I} \mu_i\right)(y)$
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(ii) \( (V_{i\ell} \mu_i)(xy) = \sup \{ \mu_i(xy^i) : i \in I \} \)
\[ \geq \sup \{ \mu_i(y) : i \in I \} \]
\[ = (V_{i\ell} \mu_i)(y) \quad \text{and} \]
\( (V_{i\ell} \mu_i)((x + z)y - xy) = \sup \{ \mu_i((x + z)y - xy) : i \in I \} \)
\[ \geq \sup \{ \mu_i(z) : i \in I \} \]
\[ = (V_{i\ell} \mu_i)(z) \]

(iii) \( x \leq y \Rightarrow (V_{i\ell} \mu_i)(x) = \sup \{ \mu_i(x) : i \in I \} \)
\[ \geq \sup \{ \mu_i(y) : i \in I \} \]
\[ = V_{i\ell} \mu_i(y) \]

Hence \( V_{i\ell} \mu_i \) is a T-fuzzy ideal of \( N \).

**Theorem (2):** An epimorphic pre-image of a T-fuzzy ideal of a PON \( N \) is a T-fuzzy ideal.

**Proof.** Let \( R \) and \( S \) be T-fuzzy ideals of a PON \( N \). Let \( f : R \rightarrow S \) be an epimorphism. Let \( v \) be a T-fuzzy ideal of \( S \) and \( \mu \) be the pre-image of \( v \) under \( f \). Then for any \( x, y, z \in R \), we have

(i) \( \mu(x - y) = (v \circ f)(x - y) \)
\[ \geq v(f(x - y)) = v(f(x) - f(y)) \]
\[ \geq T(v(f(x)), v(f(y))) \]
\[ = T((v \circ f)(x), (v \circ f)(y)) \]
\[ = T(\mu(x), \mu(y)) \]

(ii) \( \mu(xy) = (v \circ f)(xy) \)
\[ \geq v(f(xy)) = v(f(x)f(y)) \]
\[ \geq \mu(f(y)) \]
\[ = (v \circ f)(y) \]
\[ = \mu(y) \quad \text{and} \]
\( \mu((x + z)y - xy) = (v \circ f)((x + z)y - xy) \)
\[ = v(f((x + z)y - xy)) \]
\[ = v(f(yz)) \]
\[ = v(f(x)f(z)) \]
\[ \geq v(f(z)) \]
\[ = (v \circ f)(z) \]
\[ = \mu(z). \]

(iii) \( x \leq y. Then \mu(x) = (v \circ f)(x) \)
\[ = v(f(x)) \]
\[ = v(f(y)) \]
\[ = (v \circ f)(y) \]
\[ = \mu(y). \]
Hence $\mu$ is a T-fuzzy ideal of a PON $N$.

**Theorem (3):** Let $\mu$ be a T-fuzzy ideal of PON $N$ and $\mu^*$ be a fuzzy set in $N$ defined by $\mu^*(x) = \frac{\mu(x)}{\mu(1)}$

for all $x \in N$. Then $\mu^*$ is normal T-fuzzy ideal of $N$ containing $\mu$.

**Proof.** Let $\mu$ be a T-fuzzy ideal of a PON $N$. For any $x, y, z \in N$, then

(i) $\mu^*(x - y) = \frac{\mu(x - y)}{\mu(1)}$

$\geq \frac{1}{\mu(1)} T((\mu(x), (\mu(y)))$

$= T\left(\frac{1}{\mu(1)} \mu(x), \frac{1}{\mu(1)} \mu(y)\right)$

$= T\left(\mu^*(x), \mu^*(y)\right)$.

(ii) $\mu^*(xy) = \frac{\mu(xy)}{\mu(1)}$

$\geq \frac{1}{\mu(1)} (\mu(y))$

$= \mu^*(y)$ and

$\mu^*((x + z)y - xy) = \frac{\mu((x + z)y - xy)}{\mu(1)}$

$\geq \frac{1}{\mu(1)} (\mu(z))$

$= \mu^*(z)$.

(iii) $x \leq y \Rightarrow \mu^*(x) = \frac{\mu(x)}{\mu(1)}$

$\geq \frac{\mu(y)}{\mu(1)}$

$= \mu^*(y)$.

Hence $\mu^*$ is a T-fuzzy ideal of $N$. Clearly $\mu^*(1) = \frac{1}{\mu(1)} \mu(1) = 1$ and $\mu \subseteq \mu^*$.

**Lemma (1):** Let $R$ and $S$ be a PON’S and $f : R \rightarrow S$ is a homomorphism. Let $\mu$ be $f$-invariant fuzzy ideal of $R$. If $x = f(a)$, then $f(\mu)(x) = \mu(a)$ for all $a \in R$.

**Theorem (4):** Let $f : R \rightarrow S$ be an epimorphism of a PON’S $R$ and $S$. If $\mu$ is $f$-invariant T-fuzzy ideal of $R$, then $f(\mu)$ is a T-fuzzy ideal of $S$.

**Proof.** Let $a, b, c \in S$. Then there exist $x, y, z \in R$ such that $f(x) = a, f(y) = b$ and $f(z) = c$. Suppose $\mu$ is $f$-invariant T-fuzzy ideal of $R$. Then we have
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(i) \( f(\mu)(a-b) = f(\mu)(f(x) - f(y)) \)
    \[= f(\mu)f(x - y) \]
    \[= \mu(x - y) \]
    \[\geq T(\mu(x), \mu(y)) \]
    \[= T(f(\mu)(a), f(\mu)(b)). \]

(ii) \( f(\mu)(ab) = f(\mu)(f(x)f(y)) \)
     \[= f(\mu)f(xy) \]
     \[= \mu(xy) \]
     \[\geq \mu(x) \]
     \[= f(\mu)(b) \quad \text{and} \]

\[ f(\mu)((a+b)c-ab)) = f(\mu)(f(x+y)z - f(xy)) \]
\[= f(\mu)(f(x) + f(y))f(z) - f(x)f(y)) \]
\[= f(\mu)(f(x+y)f(z) - f(x)f(y)) \]
\[= \mu((x+y)z - xy) \]
\[\geq \mu(z) \]
\[= f(\mu)(c) \]

(iii) Let \( a \leq b \Rightarrow f(\mu)(a) \)
     \[= f(\mu)(f(x)) \]
     \[= \mu(x) \]
     \[\geq \mu(y) \]
     \[= f(\mu)(b). \]

Hence \( f(\mu) \) is a T-fuzzy ideal of \( S \).

**Theorem (5):** Let \( \mu \) be a T-fuzzy left ideal of PON \( N \) and \( \mu^*(x) = \mu(x) + 1 - \mu(0) \) for all \( x \in N \). Then \( \mu^* \) is a normal T-fuzzy left ideal of \( N \) containing \( \mu \), provided t-norm holds for combined translation.

**Proof:** Let \( \mu \) be a T-fuzzy left ideal of PON \( N \). We have \( \mu^*(x) = \mu(x) + 1 - \mu(0) \) for all \( x \in N \). Put \( 1 - \mu(0) = a \) then \( \mu^*(x) = \mu(x) + a \) and hence \( \mu^*(x) = \mu^T \). \( \mu^* \) is a T-fuzzy left ideal of \( N \). By definition of \( \mu^* \), \( \mu \leq \mu^* \) and \( \mu^*(0) = \mu(0) + 1 - \mu(0) \) and hence \( \mu^*(0) = 1 \). Therefore \( \mu^* \) is a normal T-fuzzy left ideal of \( N \).

**Theorem (6):** Let \( \psi \) be an imaginable fuzzy subset of partially ordered near-ring \( N \). Then \( \psi \) is a T-fuzzy left ideal of a partially ordered near-ring \( N \) if and only if the strongest fuzzy relation \( \mu_\psi \) on \( N \) is an imaginable T-fuzzy left ideal of partially ordered near-ring \( N \times N \).

**Proof:** Suppose that \( \psi \) is an imaginable T-fuzzy left ideal of PON \( N \) then obviously \( \mu_\psi \) is a T-fuzzy left ideal of a PON \( N \times N \), for any \((x_1, x_2), (y_1, y_2) \in N \times N \). Then
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\[ \mu_{\psi}(x_1, x_2) = T(\psi(x_1), \psi(x_2)) \geq T(T(\psi(x_1-x_1), \psi(y_1)), T(T(\psi(x_2-x_2), \psi(y_2))) \]

\[ = T(\mu_{\psi}(x_1-y_1, x_2-y_2), \mu_{\psi}(y_1-y_2)) \]

\[ T(\mu_{\psi}(x_1, x_2), \mu_{\psi}(x_1, x_2)) = T(T(\psi(x_1), \psi(x_1), T(\psi(x_1), \psi(x_2))) \]

\[ = T(T(\psi(x_1), \psi(x_1), T(\psi(x_2), \psi(x_2))) = T(\psi(x_1), \psi(x_2)) = \mu_{\psi}(x_1, x_2) \]

Suppose \((x_1, x_2), (y_1, y_2) \in N \times N \) and \((x_1, x_2) \leq (y_1, y_2)\) then \(x_1 \leq y_1\) and \(x_2 \leq y_2\). Therefore \(T(\psi(x_1), \psi(x_2)) \geq T(\psi(y_1), \psi(y_2))\). Hence \(\mu_{\psi}(x_1, x_2) \geq \mu_{\psi}(y_1, y_2)\). Thus \(\mu_{\psi}\) is an imaginable T-fuzzy left ideal of a partial ordered near-ring. Let \(x, y \in N\). Then

(i) \(\psi(x - y) = T(\psi(x - y), \psi(x - y)) = \mu_{\psi}(x - y, x - y) = \mu_{\psi}((x, x) - (y, y)) \geq T(\mu_{\psi}(x, x), (y, y)) = T(T(\psi(x), \psi(x)), T(\psi(y), \psi(y))) = T(\mu_{\psi}(x, x), \mu_{\psi}(y, y)) = T(T(\mu_{\psi}(x, x), \mu_{\psi}(x, y))) = T(\psi(x), \psi(y)) \)

(ii) \(\psi(xy) = T(\psi(xy), \psi(xy)) = \mu_{\psi}(xy, xy) = \mu_{\psi}((x, x)(y, y)) \geq T(\mu_{\psi}(y, y)) = T(\psi(y), \psi(y)) = \psi(y)\) and \(\psi(x) = T(\psi(x), \psi(x)) = \mu_{\psi}(x) \geq T(\mu_{\psi}(x - y, x - y), \mu_{\psi}(y, y)) = T(T(\psi(x - y), \psi(x - y)), T(\psi(y), \psi(y))) = T(\psi(x - y), \psi(y)) \)

(iii) Let \(x, y \in N\) and \(x \leq y\). then \((x, x) \leq (y, y)\)

\[ \mu_{\psi}(x, x) \geq \mu_{\psi}(y, y) \]

\[ T(\psi(x), \psi(x)) \geq T(\psi(y), \psi(y)) \]

Hence \(\psi\) is a T-fuzzy left ideal of a PON N.

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AUTHORS’ BIOGRAPHY

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