\textbf{Γ-Semigroups in which Primary Γ- Ideals are Prime and Maximal}

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Abstract: In this paper, the terms, Maximal Γ- ideal, Primary Γ-semigroup, prime Γ-ideal and simple Γ-semigroup are introduced. It is proved that if \( S \) is a Γ-semigroup containing 0 and identity with the maximal Γ-ideal \( M \). Then every non zero primary Γ-ideal is prime as well as maximal if and only if \( S \backslash M \) is a 0-simple Γ-semigroup with either 1) \( M = (S \backslash M) \Gamma a^{-1} (S \backslash M) \cup \{0\}, a \in M \) and \( \langle a \rangle \cap \langle a \rangle = 0 \) or 2) \( M \) is a 0-simple Γ-semigroup. Also it is proved that if \( S \) is a duo Γ-semigroup containing 0 and identity with the maximal Γ-ideal \( M \). Then every non zero primary Γ-ideal is prime as well as maximal if and only if \( S \backslash M \) is one of the following types 1) \( S = G \cup M \) where \( G \) is the Γ-group of units and \( M = \{a \in G : a \gamma = 0, a \in M, \gamma \in \Gamma \} \cup \{0\} \). 2) \( S \) is the union of two Γ-semigroups with 0-adjoined. Also it is proved that if \( S \) is a commutative Γ-semigroup with 0 and identity and with the maximal Γ-ideal \( M \). Suppose that every non zero primary Γ-ideal is prime or every nonzero Γ-ideal is prime. Then \( S \) satisfies one of the following conditions 1) \( S = G \cup M \) where \( G \) is the Γ-group of units in \( S \) and \( M = (a \Gamma G) \cup \{0\}, a \in M \) and \( \alpha a = 0 \) 2) \( (MI)^n = M \) for every positive integer \( n \). Furthermore if \( S \) has maximum condition on Γ-ideals then for every \( m \in M \), we have \( m \in M \Gamma e, e \) being a proper idempotent and also proved that if \( S \) is a quasi commutative Noetherian Γ-semigroup containing identity. Suppose every primary Γ-ideal in \( S \) is prime. Then the following are equivalent 1) \( S \) is cancellative, 2) \( S \) has no proper Γ-ideomponents. 3) \( S \) is a Γ-group.


Keywords: Γ-semigroup, Maximal Γ-ideal, Primary Γ-semigroup, commutative Γ-semigroup, left (right) identity, identity, Zero element, Prime Γ-ideal simple Γ-semigroup and duo Γ-semigroup.

1. INTRODUCTION


2. PRELIMINARIES

\textbf{DEFINITION 2.1:} Let \( S \) and \( \Gamma \) be any two non-empty sets. Then \( S \) is said to be a \textbf{Γ-semigroup} if there exist a mapping from \( S \times \Gamma \times S \to S \) which maps \( (a, \gamma, b) \to a \gamma b \) satisfying the condition : \( (ab)c = a(bc) \) for all \( a, b, c \in S \) and \( \alpha, \beta, \gamma \in \Gamma \).

\textbf{NOTE 2.2:} Let \( S \) be a Γ-semigroup. If \( A \) and \( B \) are two subsets of \( S \), we shall denote the set \( \{a \beta b : a \in A , b \in B \text{ and } \gamma \in \Gamma \} \) by \( \gamma \Gamma B \).

\textbf{DEFINITION 2.3:} A Γ-semigroup \( S \) is said to be \textbf{commutative Γ-semigroup} provided \( a \gamma b = b \gamma a \) for all \( a, b \in S \) and \( \gamma \in \Gamma \).

\textbf{NOTE 2.4 :} If \( S \) is a commutative Γ-semigroup then \( a \Gamma b = b \Gamma a \) for all \( a, b \in S \).
**NOTE 2.5:** Let $S$ be a $\Gamma$-semigroup and $a, b \in S$ and $a \in \Gamma$. Then $aaba$ is denoted by $(aa)b$ and consequently $aaaaab$ is denoted by $(aa)^2b$.

**DEFINITION 2.6:** A $\Gamma$-semigroup $S$ is said to be **quasi commutative** provided for each $a, b \in S$, there exists a natural number $n$ such that $aby = (by)^n a \forall y \in \Gamma$.

**NOTE 2.7:** If a $\Gamma$-semigroup $S$ is **quasi commutative** then for each $a, b \in S$, there exists a natural number $n$ such that $a\Gamma b = (b \Gamma)^n a$.

**DEFINITION 2.8:** An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **left identity** of $S$ provided $a\alpha s = s$ for all $s \in S$ and $\alpha \in \Gamma$.

**DEFINITION 2.9:** An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **right identity** of $S$ provided $s\alpha a = s$ for all $s \in S$ and $\alpha \in \Gamma$.

**DEFINITION 2.10:** An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **two sided identity** or an identity provided it is both a left identity and a right identity of $S$.

**NOTATION 2.11:** Let $S$ be a $\Gamma$-semigroup. If $S$ has an identity, let $S_1 = S$ and if $S$ does not have an identity, let $S_1$ be the $\Gamma$-semigroup $S$ with identity adjoined, usually denoted by the symbol $1$.

**DEFINITION 2.12:** An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **left zero** of $S$ provided $a\Gamma s = a$ for all $s \in S$.

**DEFINITION 2.13:** An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **right zero** of $S$ provided $s\Gamma a = a$ for all $s \in S$.

**DEFINITION 2.14:** An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **zero** of $S$ provided it is both left and right zero of $S$.

**NOTATION 2.15:** Let $S$ be a $\Gamma$-semigroup. If $S$ has a zero, let $S^0 = S$ and if $S$ does not have a zero, let $S^0$ be the $\Gamma$-semigroup $S$ with **zero adjoined**, usually denoted by the symbol $0$.

**DEFINITION 2.16:** A non empty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a **left $\Gamma$-ideal** of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

**NOTE 2.17:** A non empty subset $A$ of a $\Gamma$-semigroup $S$ is a **left $\Gamma$-ideal** of $S$ iff $S \Gamma A \subseteq A$.

**DEFINITION 2.18:** A non empty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a **right $\Gamma$-ideal** of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

**NOTE 2.19:** A non empty subset $A$ of a $\Gamma$-semigroup $S$ is a **right $\Gamma$-ideal** of $S$ iff $A\Gamma S \subseteq A$.

**DEFINITION 2.20:** A non empty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a **two sided $\Gamma$-ideal** or simply a $\Gamma$-ideal of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ imply $s\alpha a \in A, a\alpha s \in A$.

**DEFINITION 2.21:** A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a **maximal $\Gamma$-ideal** provided $A$ is a proper $\Gamma$-ideal of $S$ and is not properly contained in any proper $\Gamma$-ideal of $S$.

**DEFINITION 2.22:** A $\Gamma$-ideal $P$ of a $\Gamma$-semigroup $S$ is said to be a **prime $\Gamma$-ideal** provided $A, B$ are two $\Gamma$-ideals of $S$ and $AB \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

**DEFINITION 2.23:** A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a **semiprime $\Gamma$-ideal** provided $x \in S, x\Gamma S \Gamma x \subseteq A$ implies $x \in A$.

**DEFINITION 2.24:** If $A$ is a $\Gamma$-ideal of a $\Gamma$-semigroup $S$, then the intersection of all prime $\Gamma$-ideals of $S$ containing $A$ is called a **prime $\Gamma$-radical** or simply **$\Gamma$-radical** of $A$ and it is denoted by $\sqrt{A}$ or $\text{rad } A$.

**THEOREM 2.25 [5]:** If $A$ is a $\Gamma$-ideal of a $\Gamma$-semigroup $S$ then $\sqrt{A}$ is a semiprime $\Gamma$-ideal of $S$.

**THEOREM 2.26 [5]:** A $\Gamma$-ideal $Q$ of a $\Gamma$-semigroup $S$ is a semiprime $\Gamma$-ideal of $S$ iff $\sqrt{Q} = Q$ implies $x\Gamma S \Gamma y \subseteq A$.

**DEFINITION 2.27:** An element $a$ of a $\Gamma$-semigroup $S$ is said to be **left cancellative** provided $a \Gamma x = a \Gamma y$ for all $x, y \in S$ implies $x = y$. 
\textbf{DEFINITION 2.28:} An element \( a \) of a \( \Gamma \)-semigroup \( S \) is said to be \textit{right cancellative} provided \( x \Gamma a = y \Gamma a \) for all \( x, y \in S \) implies \( x = y \).

\textbf{DEFINITION 2.29:} An element \( a \) of a \( \Gamma \)-semigroup \( S \) is said to be \textit{cancellative} provided it is both left and right cancellative element.

\textbf{DEFINITION 2.30:} A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a \textit{left primary \( \Gamma \)-ideal} provided
1) If \( X, Y \) are two \( \Gamma \)-ideals of \( S \) such that \( X \Gamma Y \subseteq A \) and \( Y \not\subseteq A \) then \( X \not\subseteq \sqrt{A} \).
2) \( \sqrt{A} \) is a prime \( \Gamma \)-ideal of \( S \).

\textbf{DEFINITION 2.31:} A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a \textit{right primary \( \Gamma \)-ideal} provided
1) If \( X, Y \) are two \( \Gamma \)-ideals of \( S \) such that \( X \Gamma Y \subseteq A \) and \( X \not\subseteq \sqrt{A} \) then \( Y \not\subseteq \sqrt{A} \).
2) \( \sqrt{A} \) is a prime \( \Gamma \)-ideal of \( S \).

\textbf{EXAMPLE 2.32:} Let \( S = \{ a, b, c \} \) and \( \Gamma = \{ x, y, z \} \). Define a binary operation \( \cdot \) in \( S \) as shown in the following table.

\[
\begin{array}{ccc}
  \cdot & a & b & c \\
  a & a & b & c \\
  b & a & a & a \\
  c & b & c & c \\
\end{array}
\]

Define a mapping \( S \times \Gamma \times S \to S \) by \( a \cdot \gamma \cdot b = ab \), for all \( a, b \in S \) and \( \gamma \in \Gamma \). It is easy to see that \( S \) is a \( \Gamma \)-semigroup. Now consider the \( \Gamma \)-ideal \( \langle a \rangle = S^1 \Gamma a \Gamma S^1 = \{ a \} \). Let \( p \Gamma q \subseteq \langle a \rangle \), \( p \not\subseteq \langle a \rangle \) \( \Rightarrow q \not\subseteq \sqrt{\langle a \rangle} \Rightarrow (q \Gamma n^{-1} q) \subseteq \langle a \rangle \) for some \( n \in N \). Since \( b \Gamma c \subseteq \langle a \rangle \), \( c \not\subseteq \langle a \rangle \Rightarrow b \subseteq \langle a \rangle \). Therefore \( \langle a \rangle \) is left primary. If \( b \not\subseteq \langle a \rangle \) then \( (c \Gamma n^{-1} c) \not\subseteq \langle a \rangle \) for any \( n \in N \Rightarrow c \not\subseteq \sqrt{\langle a \rangle} \). Therefore \( \langle a \rangle \) is not right primary.

\textbf{DEFINITION 2.33:} A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a \textit{primary \( \Gamma \)-ideal} provided \( A \) is both left primary \( \Gamma \)-ideal and right primary \( \Gamma \)-ideal.

\textbf{DEFINITION 2.34:} A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a \textit{principal \( \Gamma \)-ideal} provided \( A \) is a \( \Gamma \)-ideal generated by a single element \( a \). It is denoted by \( J[a] = \langle a \rangle \).

\textbf{DEFINITION 2.35:} An element \( a \) of a \( \Gamma \)-semigroup \( S \) with 1 is said to be \textit{left invertible} or \textit{left unit} provided there is an element \( b \in S \) such that \( b \Gamma a = 1 \).

\textbf{DEFINITION 2.36:} An element \( a \) of a \( \Gamma \)-semigroup \( S \) with 1 is said to be \textit{right invertible} or \textit{right unit} provided there is an element \( b \in S \) such that \( a \Gamma b = 1 \).

\textbf{DEFINITION 2.37:} An element \( a \) of a \( \Gamma \)-semigroup \( S \) is said to be \textit{invertible} or a \textit{Unit} in \( S \) provided it is both left and right invertible element in \( S \).

\textbf{DEFINITION 2.38:} A \( \Gamma \)-semigroup \( S \) is said to be a \textit{simple \( \Gamma \)-semigroup} provided \( S \) has no proper \( \Gamma \)-ideals.

\textbf{DEFINITION 2.39:} An element \( a \) of a \( \Gamma \)-semigroup \( S \) is said to be a \textit{\( \Gamma \)-idempotent} provided \( a \alpha a = a \) for all \( \alpha \in \Gamma \).

\textbf{NOTE 2.40:} If an element \( a \) of a \( \Gamma \)-semigroup \( S \) is a \textit{\( \Gamma \)-idempotent}, then \( a \Gamma a = a \).

\textbf{DEFINITION 2.41:} A \( \Gamma \)-semigroup \( S \) is said to be an \textit{idempotent \( \Gamma \)-semigroup} or a \textit{band} provided every element in \( S \) is a \( \Gamma \)-idempotent.

\textbf{DEFINITION 2.42:} A \( \Gamma \)-semigroup \( S \) is said to be a \textit{globally idempotent \( \Gamma \)-semigroup} provided \( S \Gamma S = S \).

\textbf{DEFINITION 2.43:} A \( \Gamma \)-semigroup \( S \) is said to be a \textit{left duo \( \Gamma \)-semigroup} provided every left \( \Gamma \)-ideal of \( S \) is a two sided \( \Gamma \)-ideal of \( S \).

\textbf{DEFINITION 2.44:} A \( \Gamma \)-semigroup \( S \) is said to be a \textit{right duo \( \Gamma \)-semigroup} provided every right \( \Gamma \)-ideal of \( S \) is a two sided \( \Gamma \)-ideal of \( S \).

\textbf{DEFINITION 2.45:} A \( \Gamma \)-semigroup \( S \) is said to be a \textit{duo \( \Gamma \)-semigroup} provided it is both a left duo \( \Gamma \)-semigroup and a right duo \( \Gamma \)-semigroup.
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**Definition 2.46**: An element \( a \) of a Γ-semigroup \( S \) is said to be **regular** provided \( a = a x b a \) for some \( x \in S, a, b \in \Gamma \). i.e., \( a \in a \Gamma S \Gamma a \).

**Definition 2.47**: A Γ- semigroup \( S \) is said to be a **regular Γ- semigroup** provided every element is regular.

**Definition 2.48**: An element \( a \) of a Γ-semigroup \( S \) is said to be **left regular** provided \( a = a a x f x \), for some \( x \in S, a, b \in \Gamma \). i.e., \( a \in a \Gamma a \Gamma S \).

**Definition 2.49**: An element \( a \) of a Γ- semigroup \( S \) is said to be **right regular** provided \( a = a x a f a \), for some \( x \in S, a, b \in \Gamma \). i.e., \( a \in S \Gamma a \Gamma a \).

**Definition 2.50**: An element \( a \) of a Γ- semigroup \( S \) is said to be **completely regular** provided there exists an element \( x \in S \) such that \( a = a x x \beta \) for some \( a, \beta \in \Gamma \) and \( a x x = x f a \), for all \( a, \beta \in \Gamma \). i.e., \( a \in a \Gamma x \Gamma a \) and \( a \Gamma x = x \Gamma a \).

**Definition 2.51**: A Γ-semigroup \( S \) is said to be a **completely regular Γ- Semigroup** provided every element is completely regular.

**Definition 2.52**: An element \( a \) of a Γ-semigroup \( S \) is said to be **intra regular** provided \( a = a x a f a \gamma y \) for some \( x, y \in S \) and \( a, b, \gamma \in \Gamma \).

**Definition 2.53**: An element \( a \) of a Γ- semigroup \( S \) is said to be **semisimple** provided \( a \in <a> \Gamma <a> \), that is, \( <a> \Gamma <a> = <a> \).

**Definition 2.54**: A Γ-semigroup \( S \) is said to be **semisimple Γ- semigroup** provided every element is a semisimple.

**Definition 2.55**: A Γ-semigroup \( S \) is said to be a **Noetherian Γ- semigroup** provided every ascending chain of Γ-ideals becomes stationary.

**Theorem 2.56** [6]: Let \( S \) be a Γ-semigroup with identity and \( M \) be the unique maximal Γ-ideal of \( S \). If \( \forall A = M \) for some Γ-ideal of \( S \). Then \( A \) is a primary Γ-ideal.

**Theorem 2.57** [6]: If \( S \) is a duo Γ-semigroup, then the following are equivalent for any element \( a \in S \).
1) \( a \) is completely regular.
2) \( a \) is regular.
3) \( a \) is left regular.
4) \( a \) is right regular.
5) \( a \) is intra regular.
6) \( a \) is semisimple.

**Theorem 2.58**: Let \( S \) be a Γ- semigroup with identity. If (non zero, assume this if \( S \) has zero) proper prime Γ-ideals in \( S \) are maximal then \( S \) is primary Γ-semigroup.

**Proof**: Since \( S \) contains identity, \( S \) has a unique maximal Γ- ideal \( M \), which is the union of all proper Γ- ideals in \( S \). If \( A \) is a (non zero) proper Γ-ideal in \( S \) then \( \forall A = M \) and hence by theorem 2.56, \( A \) is primary Γ- ideal. If \( S \) has zero and if \( <0> \) is a prime Γ-ideal, then \( <0> \) is primary and hence \( S \) is primary. If \( <0> \) is not a prime Γ-ideal, then \( \forall <0> = M \) and hence by theorem 2.56, \( <0> \) is a primary Γ- ideal. Therefore \( S \) is a primary Γ- semigroup.

**Definition 2.59**: A Γ-semigroup \( S \) is said to be a **Γ-group** provided \( S \) has no left and right Γ-ideals.

3. **Primary Γ- Ideals Are Prime and Maximal**

**Theorem 3.1**: Let \( S \) be a Γ-semigroup containing 0 and identity with the maximal Γ-ideal \( M \). Then every nonzero primary Γ-ideal is prime as well as maximal if and only if \( S \setminus M \) is a 0-simple Γ-semigroup with either
1) \( M = (S \setminus M) \Gamma a (S \setminus M) \cup \{0\}, a \in M \) and \( <a> \Gamma <a> = 0 \)
or
\textbf{THEOREM 3.4:} Let $S$ be a duo $\Gamma$-semigroup. Then every nonzero primary $\Gamma$-ideal is prime as well as maximal if and only if $S$ is either a simple $\Gamma$-semigroup or a 0-simple extension of a simple $\Gamma$-semigroup.

\textbf{Proof:} The proof can write by using theorem 3.1.

\textbf{THEOREM 3.5:} Let $S$ be a $\Gamma$-semigroup containing identity and not containing 0. Then every primary $\Gamma$-ideal is prime as well as maximal if and only if $S$ is either a simple $\Gamma$-semigroup or a 0-simple extension of a simple $\Gamma$-semigroup.

\textbf{Proof:} The proof can write by using theorem 3.1.
\textbf{THEOREM 3.11:} Let $S$ be a quasi commutative Noetherian $\Gamma$-ideal $M$ is a prime $\Gamma$-ideal and hence $M \Gamma M$ is a prime $\Gamma$-ideal by hypothesis. Thus $M = M \Gamma M$ and hence $M = (M \Gamma)^{n+1} M$ for every natural number $n$.

\textbf{THEOREM 3.8:} Let $S$ be a $\Gamma$-semigroup containing identity and not containing 0 in which primary $\Gamma$-ideals are prime. Then $S$ is a 0-simple $\Gamma$-semigroup extention of a globally idempotent $\Gamma$-semigroup.

\textbf{Proof:} The proof of this theorem is a direct consequence of theorem 3.7.

\textbf{THEOREM 3.10:} Let $S$ be a $\Gamma$-semigroup containing 0 and identity with the maximal $\Gamma$-ideal $M$. If every nonzero primary $\Gamma$-ideal is prime, then $S$ satisfies either one of the following conditions.

1) $S = G \cup M$ where $G$ is the $\Gamma$-group of units in $S$ and $M = a \Gamma G \cup \{0\}, a \in M$ and $a \Gamma a = 0$.

2) $(M \Gamma)^{n+1} M = M$ for every natural number $n$. Furthermore if $S$ is Noetherian and quasi commutative, then for every $a \in M$, we have $a \in a \Gamma e$, $e$ being proper idempotent in $S$.

\textbf{Proof:} By theorem 3.7, if every nonzero primary $\Gamma$-ideal is prime, then either 1) $S = G \cup M$, where $G$ is the $\Gamma$-group of units in $S$ and $M = (a \Gamma G) \cup \{0\}, a \in M$ and $a \Gamma a = 0$, 2) $(M \Gamma)^{n+1} M = M$ for every natural number $n$. Suppose $S$ is a Noetherian quasi commutative $\Gamma$-semigroup with $M \Gamma M = M$. Since $M \Gamma M = M$, every $x \in M$ is of the form $a \Gamma b$ where $a, b \in M$. Suppose there exists a nonzero element $a \in M$ such that $a$ cannot be a product of itself and some element in $M$, that is, let $a \in b_1 \Gamma a_1$ where $a_1, b_1 \in M$ and $\neq a$. Then $a_1 \neq b_2 \Gamma a_2$ where $b_2, a_2 \in M$ and $\neq a_1$. Since otherwise $a_1 \neq b_1 \Gamma a_2$ and so $a \not\in a_1 \Gamma a_2$, this is a contradiction. Proceeding in this manner, we have $a_2 \neq b_3 \Gamma a_3, \ldots, a_{n+1} \neq b_{n+1} \Gamma a_{n+1}, \ldots$. Thus we obtain a strictly ascending chain of $\Gamma$-ideals $<a_1> \subseteq <a_2> \subseteq \ldots$. Since $S$ is Noetherian, this chain terminates and hence we have $a_n \in b_{n+1} \Gamma a_{n+1}$ where $a_{n+1} \in S \Gamma a_n$. This implies $a_n \in b_{n+1} \Gamma e \Gamma a_n$, this is a contradiction. Therefore there does not exist a nonzero $a \in M$ such that $a$ cannot be a product of itself and some element in $M$. We claim that for every nonzero $a \in M$, $a \in a \Gamma e, e \not\in a \Gamma e \in M$. Let us assume the contrary, that is, suppose that there exists $a \in M$ such that $a$ is not a product of a $\Gamma$-idempotent and itself. So $a \in a \Gamma b_1$ where $b_1$ is not a $\Gamma$-idempotent. Clearly $<a> \subseteq <b_1>$. Since otherwise $b_1 \not\in a \Gamma a$ and so $a \neq (a \Gamma a) \Gamma a$ which implies by theorem 2.57, $a$ is regular and hence $a$ is a product of a $\Gamma$-idempotent and itself, which is a contradiction. Thus we have a strictly ascending chain of $\Gamma$-ideals $<a_1> \subseteq <a_2> \subseteq \ldots$. Since $S$ is Noetherian, this chain terminates and hence $<a_1> = <a_2> = \ldots \ldots$ for some natural number $n$. Now we have $b_n$ is a product of an idempotent and itself, this is a contradiction. Therefore $a \in a \Gamma e, e \in e \Gamma a, e \in M$.

\textbf{THEOREM 3.9:} Let $S$ be a commutative $\Gamma$-semigroup with 0 and identity and the maximal $\Gamma$-ideal $M$. Suppose that every nonzero primary $\Gamma$-ideal is prime or every nonzero $\Gamma$-ideal is prime. Then $S$ satisfies either one of the following conditions.

1) $S = G \cup M$ where $G$ is the $\Gamma$-group of units in $S$ and $M = (a \Gamma G) \cup \{0\}, a \in M$ and $a \Gamma a = 0$.

2) $(M \Gamma)^{n+1} M = M$ for every positive integer $n$. Furthermore if $S$ satisfies maximum condition on $\Gamma$-ideals then for every $m \in M$, we have $m \in M \Gamma e, e$ being a proper idempotent.

\textbf{Proof:} The proof of this theorem is an immediate consequence of theorem 3.9.

\textbf{THEOREM 3.11:} Let $S$ be a quasi commutative Noetherian $\Gamma$-semigroup containing identity. Suppose every primary $\Gamma$-ideal in $S$ is prime. Then the following are equivalent.

1) $S$ is cancellative.

2) $S$ has no proper $\Gamma$-idempotents.

3) $S$ is a $\Gamma$-group.

\textbf{Proof:} 3) implies 1) is clear. Let $e$ be a $\Gamma$-idempotent in $S$. Let $a \in S$. Now $a \gamma e = a \gamma e \gamma e$ implies $a = a \gamma e \gamma e \gamma$ in $\Gamma$. This is true for all $a \in S, \gamma \in \Gamma$. Similarly $e \gamma a = a$. Therefore $e$ is the identity in $S$. Therefore $S$ has no proper idempotents. Therefore 1) implies 2). Assume 2). If $S$ is not a $\Gamma$-group,
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then \( S \) has a unique maximal \( \Gamma \)-ideal \( M \) and hence theorem 3.9, for every \( a \in M \), \( a = a \Gamma e \) for some proper idempotent \( e \). This is a contradiction. Therefore 2) implies 3).

**THEOREM 3.12:** Let \( S \) be a quasi commutative \( \Gamma \)-semigroup with 0 and with out identity in which every nonzero \( \Gamma \)-ideal is prime. If \( S \) is Noetherian, then every element \( x \in S \) of the form \( x = x \Gamma t, t \in S \) or \( x \Gamma x = (x \Gamma x)\Gamma e \) where \( e \) is an idempotent. Furthermore, if \( S \) is cancellative, then every \( x \in S \) is of the form \( x = x \Gamma t, t \in S \).

**Proof:** If \( S \) has no proper nonzero \( \Gamma \)-ideals, then for any nonzero \( x \in S \), \( x \Gamma S = S \). Thus \( x = x \Gamma t, t \in S \). If \( S \) has no proper \( \Gamma \)-ideals, then \( S \) is Noetherian, \( S \) contains maximal \( \Gamma \)-ideals. Suppose there exists a maximal \( \Gamma \)-ideal \( M \) such that \( M \Gamma M = 0 \). Then for any prime \( \Gamma \)-ideal \( P \), we have \( M \Gamma M \subseteq P \) and hence \( M = P \). So \( M \) is a unique nonzero \( \Gamma \)-ideal in \( S \). Then \( 0 \neq x \in M \) implies \( x \Gamma S = M \). Hence \( x = x \Gamma t \) for some \( t \in S \). If \( x \in M \), then since \( M \) is prime \( x \Gamma x \notin M \). So \( x \Gamma S = S \). Thus \( x \in x \Gamma t \) for some \( t \in S \). Now assume that \( M \Gamma M \neq 0 \) for any maximal \( \Gamma \)-ideal \( M \). Let \( x \in S \). Then since \( S \) is Noetherian \( x \Gamma S \) contained in maximal \( \Gamma \)-ideal, say \( M \).

Since \( M \Gamma M \neq 0 \), \( M \Gamma M \) is prime and hence \( M \Gamma M = M \). Then it can be easily verified as in the proof of the theorem 3.9, that \( x \Gamma x = x \Gamma x \Gamma e \) where \( e \) is a \( \Gamma \)-idempotent. Clearly if \( S \) is cancellative, then \( x \in x \Gamma t \) for some \( t \in S \).

**Conclusion:** It is proved that if \( S \) is a quasi commutative \( \Gamma \)-semigroup with 0 and without identity in which every no-zero \( \Gamma \)-ideal is prime. If \( S \) is Noetherian, then every element \( x \in S \) of the form \( x = x \Gamma t, t \in S \) or \( x \Gamma x = (x \Gamma x)\Gamma e \) where \( e \) is an idempotent. Furthermore, if \( S \) is cancellative, then every \( x \in S \) is of the form \( x = x \Gamma t, t \in S \).

**Acknowledgment:** The Principal Author is thankful to the University Grants Commission for the award of Teacher Fellowship under Faculty Development Programme for College Teachers during the Twelfth Plan Period (2012-2017) and also thankful to the authorities of Acharya Nagarjuna University, Guntur for providing facilities to undertake the research work.

**REFERENCES**


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