k- Super Lehmer-3 Mean Graphs

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Abstract: Let \( f:V(G)\rightarrow\{1,2,3,\ldots,k+p+q-1\} \) be an injective function. For a vertex labeling the induced edge labeling \( f(e=uv) \) is defined by \( f(e) = \left[ \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right] \) (or) \( \left[ \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right] \), then \( f \) is called \( k \)-super Lehmer-3 mean labeling if \( \{f(V(G))\cup\{f(e):e\in E(G)\}\} =\{k,k+1,k+2,\ldots,k+p+q-1\} \). A graph which satisfies this labeling condition is called \( k \)-super Lehmer-3 mean graph.

Keywords: Lehmer-3 mean graph, Super Lehmer-3 mean graph, k-Super Lehmer-3 mean graph, Path, Comb, Kite, Crown.

1. Introduction

A graph considered here are finite, undirected and simple. The vertex set and edge set of a graph are denoted by \( V(G) \) and \( E(G) \) respectively. For standard terminology and notations we follow Harrary[1], S Somasundaram & S.S Sandhya introduced the concept of Harmonic Mean Labeling of Graphs in [2] and its basic results was proved in [3] and [4]. We will provide a brief summary of other informations which are necessary for our present investigation.

Definition 1.1

A graph \( G=V,E \) with \( p \) vertices and \( q \) edges is called \textbf{Lehmer-3 mean graph}. If it is possible to label vertices \( x \in V \) by \( 1,2,3,\ldots,q+1 \) in such a way that each edge \( e=uv \) is labeled with \( f(e=uv) = \left[ \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right] \) (or) \( \left[ \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right] \), then the edge labels are distinct. In this case “\( f \)” is called Lehmer-3 mean labeling of \( G \).

Definition 1.2

Let \( f:V(G)\rightarrow\{1,2,\ldots,p+q\} \) be an injective function. For a vertex labeling “\( f \)” the induced edge labeling \( f(e=uv) \) is defined by \( f(e) = \left[ \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right] \) (or) \( \left[ \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right] \), then \( f \) is called Super Lehmer-3 mean labeling if \( \{f(V(G))\cup\{f(e):e\in E(G)\}\} =\{1,2,3,\ldots,p+q\} \). A graph which admits Super Lehmer-3 Mean labeling is called \textbf{Super Lehmer-3 Mean graph}.
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**Definition 1.3**

Let $E(V(G) \rightarrow \{1,2,3,\ldots,k+p+q-1\}$ be an injunctive function, for a vertex labeling the induced edge labeling $f(e=uv)$ is defined by $f(e) = \left[ \frac{f(u)^3 + f(v)^3}{f(u)^2 + f(v)^2} \right]$ (or) $\left[ \frac{f(u)^3 + f(v)^3}{f(u)^2 + f(v)^2} \right]$, then $f$ is called $k$-super Lehmer-3 mean labeling if $f(V(G) \cup \{f(e): e \in E(G)\}) = \{k, k+1, k+2, \ldots, k+p+q-1\}$. A graph which satisfies this labeling condition is called $k$-super Lehmer-3 mean graph.

2. **MAIN RESULTS**

**Theorem 2.1**

Any path is a $k$- Super Lehmer-3 mean graph.

**Proof :**

Let $P_n$ be a path of $n$ vertices $u_1, u_2, \ldots, u_n$.

We define a function $f: V(G) \rightarrow \{k, k+1, k+2, \ldots, k+p+q-1\}$ by

$f(u_i) = k + (2i - 2)$; $i = 1, 2, 3, \ldots, n$

The edges are labeled with

$f(u_iu_{i+1}) = k + (2i - 1)$; $1 \leq i \leq n - 1$,

Then we get distinct edge labels in which both $f(V(G) \cup E(G))$ gives the values from $\{k, k+1, k+2, \ldots, k+p+q-1\}$. Thus any path forms a $k$-super Lehmer 3 mean graph.

**Example 2.2**

Let us check with $k=20$ upto 5 vertices we get.

![Figure- 1](image)

**Theorem 2.3**

Any $(P_n \circ K_1)$ is a $k$-Super Lehmer-3 mean graph.

**Proof :**

Let $P_n$ be a path with $n$ vertices $K_1$ be a pendant vertices from each vertex of the path $P_n$. let the vertices of $P_n$ be $u_1, u_2, \ldots, u_n$ and that of the pendant vertices be $v_1, v_2, \ldots, v_n$.

We define a function $f: V(G) \rightarrow \{k, k+1, k+2, \ldots, k+p+q-1\}$ by

$f(u_i) = k + (4i - 4)$; $1 \leq i \leq n$

$f(v_i) = k + (4i - 2)$; $1 \leq i \leq n$.

The edge labelings are

$f(u_iu_{i+1}) = k + (4i - 3)$; $1 \leq i \leq n - 1$,

$f(u_iv_i) = k + (4i - 1)$; $1 \leq i \leq n - 1$.

Thus the union of vertices of $G$ and edges of $G$ together equals $\{k, k+1, k+2, \ldots, k+p+q-1\}$ which are all distinct and hence forms a $k$-Super Lehmer-3 mean graph.

**Example 2.4**

12-Super Lehmer-3 mean labeling pattern on $(P_4 \circ K_1)$ is given below.
Theorem 2.5
Any graph obtained by attaching $C_3$ to an end vertex of $P_n$ is a $k$-Super Lehmer-3 mean graph.

Proof:
Let $G$ be a graph obtained by attaching $C_3$ to an end vertex of $P_n$. Let the vertices of $P_n$ be $u_1, u_2, \ldots, u_n$ and the vertices of the cycle $C_3$ be $u_nv_1w$. Define a function $f: V(G) \rightarrow \{k, k+1, k+2, \ldots, k+p+q-1\}$ by

\[
f(u_i) = k + (2i-2); \quad 1 \leq i \leq n,
\]
\[
f(v) = k + 2n \quad \text{and}
\]
\[
f(w) = k + (2n+3)
\]

Thus the edges obtained are all distinct. Also $f(V(G) \cup E(G)) = \{k, k+1, k+2, \ldots, k+p+q-1\}$. Hence this graph $G$ admits a $k$-Super Lehmer-3 mean graph.

Example 2.6
We obtain a graph by giving the value of $k=100$ we get.

Theorem 2.7
$nP_m$ is a $k$-Super Lehmer-3 mean graph.

Proof:
Let $n$ be the number of graphs and $m$ be the number of vertices of each path. Let $u_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$ be the vertices of $nP_m$. The edge set is $E = \{u_{ij}u_{ij+1}/1 \leq i \leq n, 1 \leq j \leq m-1\}$. Let us define a function $f: V(nP_m) \rightarrow \{k, k+1, k+2, \ldots, k+p+q-1\}$ by

\[
f(u_{ij}) = k + (2m-1)(i-1) + (2j-2)
\]
Then the edge labels are all distinct such that \((f(V(G) \cup E(G)) = \{k, k+1, k+2, \ldots, k+p+q-1\}\) which is a k-Super Lehmer-3 mean graph.

**Example 2.8**

50-Super Lehmer-3 mean labeling of 4P5 graph is shown below.

![Image showing a graph with labels]

**Theorem 2.9**

\(P_n(P_m \square k_1)\) is a k-Super Lehmer-3 mean graph.

**Proof**:

Let \(G\) be a graph obtained from the union of two graphs \(P_n\) and \(P_m \square k_1\), consider the vertices of \(P_n\) as \(u_1, u_2, \ldots, u_n\) and the vertices of the comb \(P_m \square k_1\) be \(v_l, w_l\) where \(1 \leq l \leq m\).

We define a function \(f: V(G) \rightarrow \{k, k+1, k+2, \ldots, k+p+q-1\}\) by

- \(f(u_i) = k + (2i-2)\) ; \(1 \leq i \leq n\)
- \(f(v_l) = k + (2n-2) + (2j-1)\) ; \(j = 1, 3, 5, \ldots, 2m-1\)
- \(f(w_l) = k + (2n-2) + (2j-1)\) ; \(j = 2, 4, 6, \ldots, 2m\) where \(1 \leq l \leq m\)

Then the edge labels are all distinct such that \((f(V(G) \cup E(G)) = \{k, k+1, k+2, \ldots, k+p+q-1\}\). Thus \(P_n(P_m \square k_1)\) is a k-Super Lehmer-3 mean graph.

**Example 2.10**

A 25-Super Lehmer-3 mean labeling of \(P_3(P_6 \square k_1)\) is

![Image showing a graph with labels]
**Theorem 2.11**

P\(_n\) union graph obtained by attaching C\(_3\) to an end vertex of P\(_m\) is a k-Super Lehmer-3 mean graph.

**Proof :-**

Let G be a graph with vertices be u\(_1\), u\(_2\), ...., u\(_n\), v\(_1\), v\(_2\), ...., v\(_m\), w, x respectively.

A function defined by f: V(G)→ {k,k+1,k+2,......k+p+q-1} by

\[ f(u_i) = k + (2i-2) \quad ; \quad 1 \leq i \leq n \]

\[ f(v_j) = k + (2(n-2)+(2j-1)) \quad ; \quad 1 \leq j \leq m \]

\[ f(w) = k + (2n-2)+(2m+1) \]

\[ f(x) = k + (2n-2)+(2m+4) \]

Thus the edge labels obtained are all distinct so that (f(V(G)∪ E(G)) =\{k,k+1,k+2,......k+p+q-1\}. Hence P\(_n\) union graph obtained by attaching C\(_3\) to an end vertex of P\(_m\) is a k-Super Lehmer-3 mean graph.

**Example 2.12**

100 -Super Lehmer-3 mean labeling is

![Graph](image)

**Theorem 2.13**

The union of comb and a graph obtained by attaching C\(_3\) to an end vertex of P\(_m\) is a k-Super Lehmer-3 mean graph.

**Proof:-**

Let G be the union of graphs. Let its vertices be u\(_1\), u\(_2\), ...., u\(_n\), v\(_1\), v\(_2\), ...., v\(_n\), w\(_1\), w\(_2\), ...., w\(_m\), x, y respectively.

Let us define a function f: V(G)→ {k,k+1,k+2,......k+p+q-1} by

\[ f(u_i) = k + (4i-4) \quad ; \quad 1 \leq i \leq n \]

\[ f(v_i) = k + (4i-2) \quad ; \quad 1 \leq i \leq n \]

\[ f(w_j) = k + (4(n-2)+(2j-1)) \quad ; \quad 1 \leq j \leq m \]

\[ f(x) = k + (4n-2)+(2m+1) \]

\[ f(y) = k + (4n-2)+(2m+4) \]

The edge labels are all different and (f(V(G)∪ E(G)) =\{k,k+1,k+2,......k+p+q-1\}. Thus G forms a k-Super Lehmer-3 mean graph.
Example 2.14

1000-Super Lehmer-3 mean labeling of G is given below.

![Diagram showing 1000-Super Lehmer-3 mean labeling of G]

Theorem 2.15

\( nP_m \cup (P_l \cup k_1) \) is a k- Super Lehmer-3 mean graph.

Proof:

Let G be a graph of \( nP_m, P_l \cup k_1 \) union graphs. Let the vertices be \( u_{ij} \) where \( 1 \leq i \leq n, 1 \leq j \leq m \), \( v_p, w_p \) where \( 1 \leq p \leq l \).

We define a function \( f : V(G) \rightarrow \{k, k+1, k+2, \ldots, k+p+q-1\} \) by

\[
\begin{align*}
    f(u_{ij}) &= k + (2m-1)(i-1) + (2j-2) \\
    f(v_p) &= k + (2m-1)(n-1) + (2m-2) + (2s-1) ; s=1,3,5,7,\ldots,2l-1 \\
    f(w_p) &= k + (2m-1)(n-1) + (2m-2) + (2s-1) ; s=2,4,6,8,\ldots,2l \text{ and } 1 \leq p \leq l
\end{align*}
\]

Then the edge labels are all distinct. Thus \( (f(V(G)) \cup E(G)) = \{k, k+1, k+2, \ldots, k+p+q-1\} \) and hence \( nP_m \cup (P_l \cup k_1) \) is a k-Super Lehmer-3 mean graph.

Example 2.16

72-Super Lehmer-3 mean labeling of G is given below.

![Diagram showing 72-Super Lehmer-3 mean labeling of G]
Remark 2.17

nPₐ∪kite is a k-Super Lehmer-3 mean graph.

REFERENCES