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Geometrical Interpretation of Singular Value Decomposition(Svd) & Applications of SVD

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Abstract: *The SVD is familiar toipc in Linear Algebra and it is a powerful techinque in many matrix computations and analysis. This paper is a dicussion of the geometric structure of a matrix using SVD. In SVD matrix can be transformed from one vector space to another vector space. The SVD has fundamental importance in several differnet applications of Linear Algebra.*

Keywords: *SVD, Geometric Structure of a Matrix, Vector Space,*

1. Introduction

Singular value decomposition

For any $m \times m$matrix A, the following decomposition always exist

$$\boldsymbol{A = UD\,V^T, A \in R^{m\times m}}$$

$$\boldsymbol{U^TU = UU^T = I_m, \in R^{m\times m}}$$

$$\boldsymbol{V^TV = VV^T = I_m, V \in R^{n\times n}}$$

$\therefore U, V$ are two orthogonal matrices and $D \in R^{m\times n}$ Diagonal matrices with non- negative entries of the diagonal called singular values.

If rank of A is $r < (m, n)$ the SVD can represented in reduced form

$$A = UDV^T, A \in R^{m\times n}$$

$$U \in R^{m\times r}$$

$$V \in R^{n\times r}$$

$$D \in R^{r\times r}$$

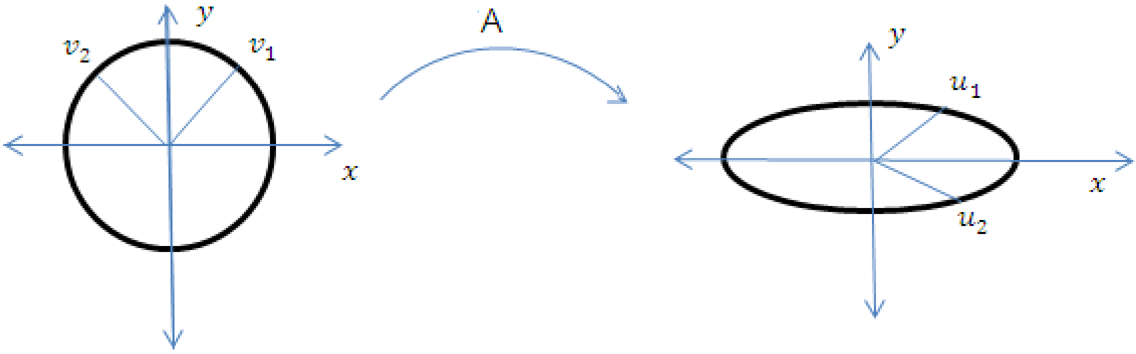
Let $A = UDV^T = \sum_{i=1}^{r} D_{ii} U_i V_i^T$

This $m \times n$ matrix $U_i V_i^T$is the product of column vector U_i and the transpose of column vectorV_i. It has rank.Thus A is weighted summation of r rank matrices

2. Geometric Interpretation of the SVD

The SVD has a nice, simple geometric interpretation. It is very easy to draw in 2-Dimension.

Let $U = [u_1 \quad u_2]$ and $V^T = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$



Let $A = UDV^T$

$$AV = (UDV^T)V$$

$$AV = UD(V^TV)$$

$$AV = UDI$$

$$AV = UD$$

Let us take the unit circle and linear transformed by A, we get an ellipse. The pre-factor of D is U, $u_1 \quad u_2$ are the major and minor axis of the ellipse. The post factor of D is V^T. The vector v_1, v_2 in V^T are the vectors that get mapped to the major and the minor axis. In 3-D the sphere is transformed to ellipsoid.

Another way to say sum of this is the set $Av_1 = \sigma_i u_i$

Which you can see by this 2-D example

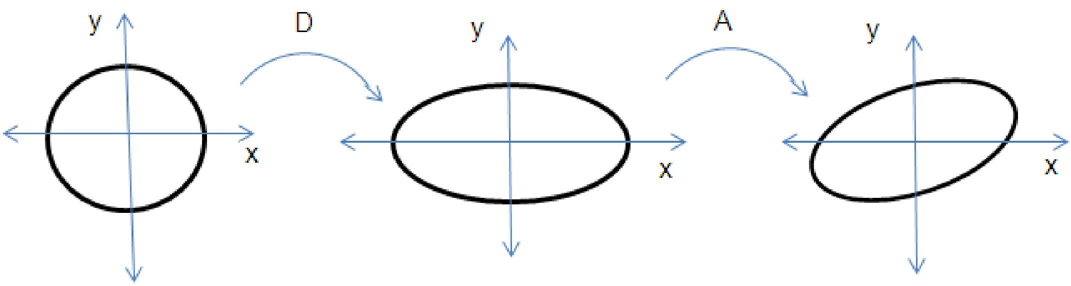
$$\begin{bmatrix} V_1^T \\ v_2^T \end{bmatrix} v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$UD \begin{bmatrix} V_1^T \\ v_2^T \end{bmatrix} v_1 = UD$$

$$(UDV^T)V_1 = UD \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Av_1 = UD \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore UD \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sigma_i u_i$$



Above discussion extended to in general mostly. If A has full rank now the singular value of A are the lengths of the n principle semi axis of the ellipsoid the length are$\sigma_1 \sigma_2 \ldots \sigma_n$ the n left singular vector of the A are the directions $u_1, u_2, .. u_n$ aligned the n semi axis of ellipsoid. The n right singular vectors of A are the directions $v_1, v_2, \ldots v_n$ in sphere which the matrix A is transformed into the semi axis of the ellipsoid.

A rank deficient matrix is one whose range is a sub space of R^3 so it maps the sphere to flat ellipse rather than ellipsoid.

$$Let\ A = [u_1 \quad u_2 \quad u_3] \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

Here one of the singular value is zero, u_3 is normal to the ellipse.

Let $V^T v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $D \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3. APPLICATION

3.1Rank- deficient least square

The linear system is $Ax = y$

$$x = A^{-1} y$$

$$x = (UDV^T)^{-1} y$$

$$x = (V^T)^{-1} D^{-1} U^{-1} y$$

$$x = (V^{-1})^T D^{-1} U^{-1} y \ [\because U \text{ is orthogonal}]$$

$$X = (VD^{-1}U^T) b$$

Where $D^{-1} = diag(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \dots, \frac{1}{\sigma_n})$

The above process is simplest but not fastest. The process is facts if the singular values are zero. If the matrix A is singular the this work.

How to solve

A is a $n \times n$ square matrix

And rank of A is $r > n$

The system will not have unique solution if $Ax^* - y$ is minimum

$A(x^* + b) - y$for any vector $b \in$ null (A) [Ab=0]

This system is like over and in determined SVD we can solve easily.

First we define residual. Sum of the square of the residual is minimum by least square method. Now to minimize the SVD of A and if then transfer it by U^T

$$\|Ax - y\|^2 = \|UDV^T x - y\|^2$$

$$= \|IUDV^T x - Iy\|^2$$

$$= \|UU^T UDV^T x - UU^T y\|^2$$

$$= \|U\| \|IDV^T x - U^T y\|^2$$

$$= \|DV^T x - U^T y\|^2$$

$$= \left\| \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^T - \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} y \right\|^2$$

$$= \left\| [D \quad 0] V^T x - u_1^T\, y \right\|^2 + \left\| u_2^T y \right\|^2$$

the first term can be made zero. The second term is residual

if we let $b = v^T x$ and $k = u_1^T y$

Than split b into $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\therefore \begin{bmatrix} D_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = k$$

$$D_1 b_1 = k$$

Now we will go for the minimizing solution than $x = V_1 b_1$

Briefly$x = V D_1^{-1} k$

$$x = V_1 D_1^{-1} U_1^T y$$

4. CONCLUSION

This paper discussed about geometric interpretation of SVD and solution of linear system with rank deficiency. This work can be extended to full rank of the matrix A and constructing least square problem.

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