Some Results on Fixed Points of Nonlinear Contraction in Metric Space

Dr. Durdana Lateef
Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawwarah, 41411, Kingdom of Saudi Arabia.

Abstract: In this paper, I have generalized the result of Sayyed and Sayyed [17] in metric spaces by replacing the containment condition and giving the shorter proof than of the authors in the main result.

Keywords and Phrases: Common fixed point, Compatible mapping, Property (E.A.), Common property (E.A.), Occasionally weakly compatible maps, Coincidence points.

1. INTRODUCTION


2. PRELIMINARIES

Throughout this paper (X, d) is a metric space which is denoted by X.

Definition 2.1: [Jungck and Rhoades [13]]. Let A and S be selfmaps of a set X. If Au = Su = ω (say), ω ∈ X, for some u in X, then u is called a coincidence point of A and S and the set of coincidence points of A and S is denoted by C(A, S), and ω is called a point of coincidence of A and S.

Definition 2.2: Let A, B, S and T be self maps of a set X. If u ∈ C(A,S) and v ∈ C(B,T) for some u,v ∈ X and Au = Su = Bu = Tv = z (say), then z is called a common point of coincidence of the pairs (A, S) and (B, T).

Definition 2.3: The pair (A, S) is said to be

(I) Satisfy property (E.A.) [1] if there exists a sequence \{x_n\} in X such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \) for some t in X.

(II) Copatible [11] if \( \lim_{n \to \infty} d(Ax_n, Sx_n) = 0 \), for some t in X whenever \{x_n\} is a sequence in X such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \).

(III) Weakly compatible [12], if they commute at their coincidence point.

(IV) Occasionally weakly compatible (owc) [3,5,6] if ASx = SAx for some \( x \in C(A,S) \)

Remark 2.4

(I) Every compatible pair is weakly compatible but its converse need not be true [12].
Definition 2.5: [14] Let \((X, d)\) be a metric space and \(A, B, S\) and \(T\) be four selfmaps on \(X\). The pairs \((A, S)\) and \((B, T)\) are said to satisfy common property \((E.A.)\) if there exists two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n\) for some \(t\) in \(X\).

Remark 2.6: Let \(A, B, S\) and \(T\) be self maps of a set \(X\). If the pairs \((A, S)\) and \((B, T)\) have common point of coincidence in \(X\) then \(C(A, S) \neq \emptyset\) and \(C(B, T) \neq \emptyset\). But converse is not true.

Example 2.7: Let \(X = [0, \infty)\) with usual metric and \(A, B, S\) and \(T\) self maps on \(X\) and defined by \(Ax = 1 - x^2; Sx = 1 - x; Bx = \frac{1}{2} + x^2; Tx = \frac{1 + x}{2}\) for all \(x \in X\).

It is easy to observe that \(C(A,S) = \{0,1\}\) and \(C(B,T) = \left\{0, \frac{1}{2}\right\}\) but the pairs \((A, S)\) and \((B, T)\) not having common point of coincidence.

Remark 2.8: The converse of the remark 2.6 is true provided it satisfies inequality (3.1). This is given as in proposition 3.1 in section III.

Preposition 2.9: [2] Let \(A\) and \(S\) be two self maps of a set \(X\) and the pair \((A, S)\) satisfies occasionally weakly compatible (owc) condition. If the pairs \((A, S)\) and \(X\) have unique point of coincidence \(Ax = Sx = z\) then \(z\) is the unique common fixed point of \(A\) and \(S\).

Proof: To be given \(Ax = Sx = \{z\}\) (say) for any \(x \in C(A,S)\)

Since the pair \((A, S)\) satisfies the property owc, therefore

\[Az = ASx = SAx = Sz\] implies that \(z \in C(A, S)\)

From (2.1), we get \(Az = Sz = z\). Hence proposition follows.

In 1996, Tas et al. [17] proved the following.

Theorem 2.10: Let \(A, B, S\) and \(T\) be self maps of a complete metric space \((X,d)\) such that \(A(X) \subseteq T(X)\) and \(B(X) \subseteq S(X)\) and satisfying the inequality.

\[
[d(Ax, By)]^2 \leq C_1 \max \{[d(Sx, Ax)]^2, [d(Ty, By)]^2, [d(Sx, Ty)]^2\} + C_2 \max \{d(Sx, Ax)d(Sx, By), d(Ty, Ax)d(Ty, By)\} + C_3 d(Sx, By)d(Ty, Ax)
\]

for all \(x, y \in X\), where \(C_1 + C_3, C_2, C_3 \geq 0, C_1 + 2C_2 < 1, C_1 + C_3 < 1\). Further, assume that the pairs \((A, S)\) and \((B, T)\) are compatible on \(X\). If one of the mappings \(A, B, S\) and \(T\) is continuous then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

3. MAIN RESULTS

Proposition 3.1. Let \(A, B, S\) and \(T\) be self maps of a metric space \((X, d)\) and satisfying the inequality.

\[
d(Ax, By) \leq k \max \left\{ \frac{d(Sx, Ax)[1+d(Ty, By)]}{1+d(Sx, Ty)}, \frac{d(Ty, By)[1+d(Sx, Ax)]}{1+d(Ty, Ax)}, \frac{d(Sx, Ax)[1+d(Sx, Ty)]}{1+d(Ty, By)}, \frac{d(Sx, By)[1+d(Ty, Ax)]}{1+d(Sx, Ty)} \right\}
\]

\[
\frac{1}{2}[d(Sx, Ax)+d(Ty, By)] + \frac{1}{2}[d(Sx, By)+d(Ty, Ax)], d(sx, Ty) \}
\]

(3.1)
Some Results on Fixed Points of Nonlinear Contraction in Metric Space

for all \( x, y \in X \), where \( k \geq 0 \) and \( k < 1 \). Then the pairs \((A, S)\) and \((B, T)\) have common point of coincidence in \( X \) if and only if \( C(A, S) \neq \emptyset \) and \( C(B, T) \neq \emptyset \).

**Proof:** If part: It is trivial

Only if part: Assume \( C(A, S) \neq \emptyset \) and \( C(B, T) \neq \emptyset \).

Then there is a \( u \in C(A, S) \) and \( v \in C(B, T) \) such that

\[
Au = Su = p \quad \text{ (say)}
\]

\[
Bu = Tv = q \quad \text{ (say)}
\]

on taking \( x = u \) and \( y = v \) in (3.1), we get

\[
d(Au, Bv) \leq k \max \left\{ \frac{d(Su, Au)[1 + d(Tv, Bv)]}{1 + d(Su, Tv)}, \frac{d(Tv, Bv)[1 + d(Su, Au)]}{1 + d(Tv, Bv)}, \frac{d(Su, Au)[1 + d(Su, Tv)]}{1 + d(Su, Tv)} \right\} + \frac{1}{2} \left[ d(Su, Bv) + d(Tv, Au) \right] \cdot d(Su, Tv)
\]

Using (3.2) and (3.3), we get

\[
d(p, q) \leq k d(p, q), \text{ a contradiction. Thus } p = q
\]

Therefore \( A, B, S \) and \( T \) have common point of coincidence in \( X \).

In the proposition (2.1) of Babu et al. [9], we can obtain some more conclusions of in their paper. Therefore our result improves and strengthen proposition 3.1 and subsequent theorems in metric spaces.

**Proposition 3.2:** Let \( A, B, S \) and \( T \) be four self maps of a metric space \((X, d)\) satisfying the inequality (3.1). Suppose that either

(i) \( B(X) \subseteq S(X) \), the pair \((B, T)\) satisfies property (E.A.) and \( T(X) \) is a closed subspace of \( X \); or

(ii) \( A(X) \subseteq T(X) \), the pair \((A, S)\) satisfies property (E.A) and \( S(X) \) is a closed subspace of \( X \) holds.

Then the pair \((A, S)\) and \((B, T)\) are satisfies the common property (E.A), also both the pairs \((A, S)\) and \((B, T)\) have common point of coincidence in \( X \).

We have shorten the proof of theorem 2.2 of [9] by relaxing many lines:

**Theorem 3.3:** (Improved version of theorem 2.2of[9])

Let \( A, B, S \) and \( T \) be satisfying all the conditions given in proposition 3.2 with the following additional assumption.

The pairs \((A, S)\) and \((B, T)\) are owc on \( X \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** By proposition 3.2 the pairs \((A, S)\) and \((B, T)\) have common point of coincidence. Therefore there is \( u \in C(A, S) \) and \( v \in C(B, T) \) such that

\[
Au = Su = z \quad \text{(say)} = Bu = Tv
\]

Now, we show that \( z \) is unique common point of coincidence of the pairs \((A, S)\) and \((B, T)\).

Let if possible \( z' \) is another point of coincidence of \( A, B, S \) and \( T \). Then there is \( u' \in C(A, S) \) and \( v' \in C(B, T) \) such that

\[
Au' = Su' = z' \quad \text{(say)} = Bu' = Tv'
\]

International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Putting \( x = u \) and \( y = v' \) in inequality (3.1), we have
\[
d(Au, Bv') \leq k \max \left\{ \frac{d(Su, Au)}{1 + d(Su, Tv')} + \frac{d(Tv', Bv')}{1 + d(Tv', Tv')} \, d(Tv', Bv'), \frac{d(Su, Au)}{1 + d(Su, Tv')} + \frac{d(Tv', Bv')}{1 + d(Tv', Tv')} \, d(Tv', Bv') \right\}.
\]

Now using (3.4) and (3.5), we get
\[
d(z, z') \leq k \, d(z, z'),
\]
and arrive at a contradiction. Hence \( z = z' \) and we have \( C(A, S) = \{ z \} = C(B, T) \). By proposition 2.9, \( z \) is the unique common fixed point of \( A, B, S \) and \( T \) in \( X \).

**Remark 3.4:** Proposition 2.5 of [9] and theorem 2.6 of [9] are remain true, if we replace completeness of \( S(X) \) and \( T(X) \) by the completeness of \( S(X) \cap T(X) \) in \( X \). For this we have given an example 2.7 in the following manner without proof.

Now we rewriting the proposition 2.5 and theorem 2.6 of [9]

**Proposition 3.5:** Let \( A, B, S \) and \( T \) be four self maps of a metric. Space \( (X, d) \) satisfying the inequality (3.1) of proposition 3.1. Suppose that \( (A, S) \) and \( (B, T) \) satisfy a common property \( (E.A) \) and \( S(X) \cap T(X) \) are closed subset of \( X \), then \( A, B, S \) and \( T \) have unique common point of coincidence. Theorem 3.6. In addition to the above proposition 3.5 on \( A, B, S \) and \( T \), if both the pairs \( (A, S) \) and \( (B, T) \) are owc mapson \( X \), then the point of coincidence is a unique common point of \( A, B, S, T \).

**REFERENCES**

Some Results on Fixed Points of Nonlinear Contraction in Metric Space


