Approximation of Alternating Series using Correction Function and Error Function

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Abstract: In this paper we give a rational approximation of an alternating series using remainder term of the series. For that we shall introduce a correction function to the series. The correction function plays a vital role in series approximation. Using correction function we shall deduce an error function to the series.

Keywords: Correction function, error function, remainder term, alternating series, Madhava series, rational approximation.

1. INTRODUCTION

In 14th century, the Indian mathematician Madhava gave an approximation of the pi series using remainder term of the series.

Madhava series is,
\[
C = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \cdots + \frac{(-1)^{n-1} 4d}{2n-1} + \frac{(-1)^n 4d(2n)/2}{(2n)^2+1},
\]
where C is the circumference of a circle of diameter d. Here the remainder term is \((-1)^n 4d G_n\) where \(G_n = \frac{(2n)/2}{(2n)^2+1}\) is the correction function. The introduction of the correction term gives a better approximation of the series.

2. METHOD

APPROXIMATION OF THE ALTERNATING SERIES \(\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n(n+1)(n+2)}\)

The alternating series \(\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n(n+1)(n+2)}\) satisfies the conditions of alternating series test and so it is convergent.

If \(R_n\) denotes the remainder term after \(n\) terms of the series, then
\[R_n = (-1)^n G_n\] where \(G_n\) is the correction function after \(n\) terms of the series

Theorem:

The correction function for the alternating series \(\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n(n+1)(n+2)}\) is
\[G_n = \frac{1}{2n^3 + 9n^2 + 2n + 3}\]

Proof:

If \(G_n\) denotes the correction function after \(n\) terms of the series, then we have \(G_n + G_{n+1} = \frac{1}{(n+1)(n+2)(n+3)}\)
The error function is \( E_n = G_n + G_{n+1} - \frac{1}{(n+1)(n+2)(n+3)} \)

For \( r_1, r_2, r_3 \in \mathbb{R} \) and for any fixed \( n \),

Let \( G_n (r_1, r_2, r_3) = \frac{1}{2n^3 + 12n^2 + 22n + 12 - (r_1n^2 + r_2n + r_3)} \)

Then the error function is

\[ E_n (r_1, r_2, r_3) = G_n (r_1, r_2, r_3) + G_{n+1}(r_1, r_2, r_3) - \frac{1}{(n+1)(n+2)(n+3)} \]

is a rational function of \( r_1, r_2 \) and \( r_3 \).

ie \( E_n(r_1, r_2, r_3) = \frac{N_n(r_1, r_2, r_3)}{D_n(r_1, r_2, r_3)} \)

\( D_n(r_1, r_2, r_3) \approx 4n^3 \) is a maximum for large \( n \).

\[ |N_n(r_1, r_2, r_3)| \text{ is minimum for } r_1 = 3, r_2 = \frac{15}{2}, r_3 = \frac{15}{4} \]

So \( |E_n(r_1, r_2, r_3)| \text{ is minimum for } r_1 = 3, r_2 = \frac{15}{2}, r_3 = \frac{15}{4} \)

Thus for \( r_1 = 3, r_2 = \frac{15}{2}, r_3 = \frac{15}{4} \) we have both \( G_n \) and \( E_n \) are functions of a single variable \( n \).

That is the correction function for the series \( \Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)(n+2)} \) is

\[ G_n = \frac{1}{2n^3 + 9n^2 + \frac{29}{2}n + \frac{33}{4}} \]

The absolute value of the error function is

\[ |E_n| = \frac{\bigg[ 27n^2 + 27n + \frac{423}{4} \bigg]}{\bigg[ 2n^3 + 9n^2 + \frac{29}{2}n + \frac{33}{4} \bigg]\bigg[ 2n^3 + 15n^2 + \frac{77}{2}n + \frac{135}{4} \bigg](n+1)(n+2)(n+3)} \]

Hence the theorem.

3. RESULTS AND DISCUSSIONS

For the series \( \Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)(n+2)} \),

(1) The correction function is \( G_n = \frac{1}{2n^3 + 9n^2 + \frac{29}{2}n + \frac{33}{4}} \)

(2) The magnitude of error function is

\[ |E_n| = \frac{\bigg[ 27n^2 + 27n + \frac{423}{4} \bigg]}{\bigg[ 2n^3 + 9n^2 + \frac{29}{2}n + \frac{33}{4} \bigg]\bigg[ 2n^3 + 15n^2 + \frac{77}{2}n + \frac{135}{4} \bigg](n+1)(n+2)(n+3)} \]

(3) Clearly \( G_n < \frac{1}{(n+1)(n+2)(n+3)} \), the absolute value of the \( (n+1) \)th term.

4. CONCLUSION

The correction function and error function play a vital role in series approximation. We can improve the accuracy of the sum of the series using these functions.

REFERENCES

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