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General Properties of Strongly Magic Squares

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Abstract: *Magic squares have been known in India from very early times. The renowned mathematician Ramanujan had immense contributions in the field of Magic Squares. A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. The paper discuss about a well-known class of magic squares; the strongly magic square. The strongly magic square is a magic square with a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. In this paper a generic definition for Strongly Magic Squares is given. Some interesting properties of Strongly Magic Squares are briefly described.*

Keywords: *Magic squares, Magic constant,Strongly Magic Squares ,Product of magic squares.*

1. Introduction

Magic squares date back in the first millennium B.C.E in China [1], developed in India and Islamic World in the first millennium C.E, and found its way to Europe in the later Middle Ages [2] and to sub-Saharan Africa not much after [3]. Magic squares generally fall into the realm of recreational mathematics [4, 5], however a few times in the past century and more recently, they have become the interest of more-serious mathematicians. Srinivasa Ramanujan had contributed a lot in the field of magic squares. Ramanujan's work on magic squares is presented in detail in Ramanujan's Notebooks [6]. A normal magic square is a square array of consecutive numbers from $1 \ldots n^2$ where the rows, columns, diagonals and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square has a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant [7]. There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possess advanced mathematical properties.

2. Notations and Mathematical Preliminaries

2.1 Magic Square

A magic square of order n over a field R where R denotes the set of all real numbers is an nth order matrix $[a_{ij}]$ with entries in R such that

$$\sum_{j=1}^{n} a_{ij} = \rho \qquad for \;\; i = 1,2,\ldots\ldots n \qquad \ldots\ldots\ldots.(1)$$

$$\sum_{j=1}^{n} a_{ji} = \rho \qquad for \;\; i = 1,2,\ldots\ldots n \qquad \ldots\ldots\ldots.(2)$$

$$\sum_{i=1}^{n} a_{ii} = \rho \;, \;\; \sum_{i=1}^{n} a_{i,n-i+1} = \rho \qquad \ldots\ldots\ldots.(3)$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant. [8]

2.2 Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$.

2.3 Strongly magic square (SMS): Generic Definition

A strongly magic square over a field R is a matrix $[a_{ij}]$ of order $n^2 \times n^2$ with entries in R such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \;\; for \;\; i = 1,2, \ldots .. n^2 \qquad \ldots \ldots \ldots \ldots .. \ldots .(4)$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \;\; for \;\; i = 1,2, \ldots .. n^2 \qquad \ldots \ldots \ldots \ldots \ldots \ldots ...(5)$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho \, , \qquad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \qquad \ldots \ldots \ldots .(6)$$

$$\sum_{l=0}^{n-1} \sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \;\; for \;\; i,j = 1,2, \ldots .. n^2 \qquad \ldots \ldots \ldots . \ldots .(7)$$

$$where\ the\ subscripts\ are\ congruent\ modulo\ n^2$$

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal & co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as ${M_{0C}}^{(n)} or {M_{0R}}^{(n)}$ and ρ is the magic constant.

Note: The n^{th} order sub square sum with k column gaps or k row gaps is generally denoted as ${M_{kC}}^{(n)}$ or ${M_{kR}}^{(n)}$ respectively.

2.4 Notations

1. Z^+ denotes the set of all positive real integers.
2. R denotes the set of all real numbers.
3. SMS *denotes the strongly magic squares*

3. Propositions and Theorems

Proposition 3.1

Let $A = \lfloor a_{ij} \rfloor$ where $1 \leq i, j \leq n^2$ be a strongly magic square of order n^2, *then*

$$\sum_{j=m}^{m+n-1} a_{ij} - \sum_{j=m}^{m+n-1} a_{i+n,j} = 0 \;\; ; \quad \text{for some} \;\; m \in Z^+$$

Proof:

$$\sum_{j=m}^{m+n-1} a_{ij} = a_{im} + a_{i,m+1} + \cdots + a_{i,m+n-1}$$

From the definition of SMS ,

$a_{i,m} + a_{i,m+1} + \cdots + a_{i,m+n-1} + a_{i+1,m} + a_{i+1,m+1} + \cdots a_{i+n-1,m} + a_{i+n-1,m+1} + \cdots \ a_{i+n-1,m+n-1} = \rho$ $\rightarrow$ (Eq. 3.1)

Also

$a_{i+1,m} + a_{i+1,m+1} + \cdots a_{i+n-1,m+n-1} + a_{i+n,m} + a_{i+n,m+1} + \cdots a_{i+n,m+n-1} = \rho$ $\rightarrow$ (Eq.3.2)

Thus equating the both the equations (3.1) and (3.2)

$a_{im} + a_{i,m+1} + \cdots + a_{i,m+n-1} = a_{i+n,m} + a_{i+n,m+1} + \cdots a_{i+n,m+n-1}$

Thus $\sum_{j=m}^{m+n-1} a_{ij} - \sum_{j=m}^{m+n-1} a_{i+n,j} = 0\ ;\ \ for\ some\ m \in\ Z^{+}$

Proposition 3.2

The sum of the corner elements of a SMS is the magic constant.

Proof:

Its an immediate consequence of the definition of SMS

Proposition 3.3

Let $[A]$ be a strongly magic square with order $n^2 \times n^2$ and $\rho(A) = p$, then there exists another strongly magic square $[B]$ of order $n^2 \times n^2$ with $\rho(B) = q$

Proof:

Let $[A] = [a_{ij}]_{n^2 \times n^2}$ for $i, j = 1,2,3 \ldots \ldots . n^2$ such that $a_{ij} \in R$

Define a square matrix $B = [b_{ij}]_{n^2 \times n^2}$ in such a way that

$$b_{ij} = \frac{p+q}{n^2} - a_{ij} \qquad \text{for} \qquad i, j = 1,2,3 \ldots \ldots . n^2$$

Now, sum of ith row element of

$$B = \sum_{j=1}^{n^2} b_{ij}$$

$$= \sum_{j=1}^{n^2} \left(\left(\frac{p+q}{n^2} \right) - a_{ij} \right)$$

$$= \frac{n^2(p+q)}{n^2} - \sum_{j=1}^{n^2} a_{ij}$$

since

$$\sum_{j=1}^{n^2} a_{ij} = p$$

Therefore

$$p + q - p = q$$

Similar computation holds for column elements

For diagonal elements

$$\sum_{i=1}^{n^2} b_{ii} = \sum_{i=1}^{n^2}\left(\left(\frac{p+q}{n^2}\right) - a_{ii}\right)$$

$$= p+q-\sum_{i=1}^{n^2} a_{ii}$$

$$= q$$

Similar computation holds for co-diagonal elements

Now for the $n \times n$ subsqaures

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} b_{i+k,j+l} = \sum_{l=0}^{n-1}\sum_{k=0}^{n-1}\left(\left(\frac{p+q}{n^2}\right) - a_{i+k,j+l}\right)$$

$$= q$$

Proposition 3.4

Let A be a SMS of order n with $\rho(A) = a$, then $B = \left[A - \frac{a}{n}U\right]$ is also a SMS with $\rho(B) = 0$ where $U = \{1\ , for\ every\ i,j = 1,2,\ldots n\}$

Proof:

$Let\ A = [a_{ij}]$ and $B = [b_{ij}] = [\ a_{ij} - \frac{a}{n}]$

The sum of the i th row elements are given by

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} [a_{ij} - \frac{a}{n}]$$

$$= \sum_{j=1}^{n} a_{ij} - a$$

$$= a - a = 0 \qquad \left(\text{Since } \sum_{j=1}^{n} a_{ij} = \rho(A) = a\right)$$

Similarly we can calculate the sum of the column elements

For the sum of the diagonal elements;

$$\sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n} [a_{ii} - \frac{a}{n}]$$

$$= \sum_{i=1}^{n} a_{ii} - a$$

$$= a - a = 0$$

For the sum of the co-diagonal elements;

$$\sum_{i=1}^{n} b_{i,n-i+1} = \sum_{i=1}^{n} [a_{i,n-i+1} - \frac{a}{n}]$$

$$= \sum_{i=1}^{n} a_{i,n-i+1} - a$$

$$= a - a = 0.$$

For the sum of the $n \times n$ sub square elements $M_{0C}{}^{(n)}$;

$$\sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} b_{i+k,j+l} = \sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} [a_{i+k,j+l} - \frac{a}{n}]$$

$$= \sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} [a_{i+k,j+l}] - a = 0.$$

Proposition 3.5

Let A be a SMS of order n with $\rho(A) = a$, then $B = \left[A - \frac{a}{\sqrt{n}} U\right]$ is also a SMS with $\rho(B) = -(\sqrt{n} - 1)\, \rho(A)$ where $U = \{1, for\ every\ i,j = 1,2, \ldots n\}$

Proof:

Let $A = \lfloor a_{ij} \rfloor$ and $B = [b_{ij}] = [\, a_{ij} - \frac{a}{\sqrt{n}}]$

Sum of the ith row elements is given by

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} [a_{ij} - \frac{a}{\sqrt{n}}]$$

$$= \sum_{j=1}^{n} a_{ij} - n.\frac{a}{\sqrt{n}}$$

$$= a - \sqrt{n}\, a \quad = \quad -a(\sqrt{n} - 1).$$

$$= -\rho(A)(\sqrt{n} - 1)\,.$$

A similar computation holds for column sum also.

For the sum of the diagonal elements,

$$\sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n} [a_{ii} - \frac{a}{\sqrt{n}}]$$

$$= \sum_{i=1}^{n} a_{ii} - n.\frac{a}{\sqrt{n}}$$

$$= -a(\sqrt{n} - 1).$$

$$= -\rho(A)(\sqrt{n} - 1)$$

For the sum of the co-diagonal elements,

$$\sum_{i=1}^{n} b_{i,n-i+1} = \sum_{i=1}^{n} [a_{i,n-i+1} - \frac{a}{\sqrt{n}}]$$

$$= \sum_{i=1}^{n} a_{i,n-i+1} - n.\frac{a}{\sqrt{n}}$$

$$= -a(\sqrt{n} - 1).$$

$$= -\rho(A)(\sqrt{n} - 1)$$

For the sum of the sub-square elements,

$$\sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} b_{i+k,j+l} = \sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} [a_{i+k,j+l} - \frac{a}{\sqrt{n}}]$$

$$= \sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} [a_{i+k,j+l}] - n \times \frac{a}{\sqrt{n}}$$

$$= -a(\sqrt{n} - 1)$$

$$= -\rho(A) \times (\sqrt{n} - 1)$$

Proposition 3.6

Let A be a SMS of order n with $(A) = a$, then $B = \left[A + \frac{a}{\sqrt{n}} U\right]$ is also an SMS with

$\rho(B) = (\sqrt{n} + 1)\rho(A)$, where $U = \{1\ , for\ every\ i,j = 1,2, \dots n\}$

Proof:

Proceeding as in Proposition 3.5, we will get the required result.

Proposition 3.7

Let A be a SMS of order n with $\rho(A) = a$, then the product $B = \left[A.\frac{a}{\sqrt{n}} U\right]$ is also a SMS with

$\rho(B) = \frac{a^2}{\sqrt{n}}$,where $U = \{1\ , for\ every\ i,j = 1,2, \dots n\}$

Proof:

Let $A = [a_{ij}]$ and $B = [b_{ij}] = [\ a_{ij} . \frac{a}{\sqrt{n}}]$

Sum of the i^{th} row elements is given by

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} [a_{ij} \times \frac{a}{\sqrt{n}}]$$

$$= \sum_{j=1}^{n} [a_{ij]} \times \frac{a}{\sqrt{n}}$$

$$= \frac{a^2}{\sqrt{n}}$$

A similar computation holds for column sum also.

For the sum of the diagonal elements,

$$\sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n} [a_{ii} \cdot \frac{a}{\sqrt{n}}]$$

$$= \sum_{i=1}^{n} [a_{ii}] . \frac{a}{\sqrt{n}}$$

$$= \frac{a^2}{\sqrt{n}}$$

For the sum of the co-diagonal elements,

$$\sum_{i=1}^{n} b_{i,n-i+1} = \sum_{i=1}^{n} [a_{i,n-i+1} \cdot \frac{a}{\sqrt{n}}]$$

$$= \sum_{i=1}^{n} [a_{i,n-i+1}] . \frac{a}{\sqrt{n}}$$

$$= \frac{a^2}{\sqrt{n}}$$

For the sum of the sub-square elements,

$$\sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} b_{i+k,j+l} = \sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} [a_{i+k,j+l} \cdot \frac{a}{\sqrt{n}}]$$

$$= \sum_{l=0}^{\sqrt{n}-1} \sum_{k=0}^{\sqrt{n}-1} [a_{i+k,j+l}] . \frac{a}{\sqrt{n}}$$

$$= \frac{a^2}{\sqrt{n}}$$

Proposition 3.8

Let A be a SMS of order n with $\rho(A) = a$, then $B = \left[A / \frac{a}{\sqrt{n}} U\right]$ is also a SMS with $\rho(B) = \sqrt{n}$ where $U = \{1 \text{ }, for\ every\ i, j = 1,2, \dots n\}$. Here the 0 matrix is excluded.

Proof:

Proceeding as in Proposition 3.7, the required result can be obtained.

4. Conclusion

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different mathematical courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described which can be used to explore new horizons. Certainly more can be done in the context of linear algebra.

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