Complementary Colour Transversal Vertex Covering Set

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Abstract: In this paper we introduce new concepts namely Complementary Colour Transversal Vertex Covering Set (CCTVC Set) and Complementary Colour Transversal Vertex Covering Number (CCTVC Number) of a graph. If G is a graph then this number is denoted as $\alpha^c_c(G)$. We have also observed that $\alpha^c_c(G) = \alpha_0(G)$ or $\alpha^c_c(G) = \alpha_0(G) + 1$ for any graph G. We proved several theorems regarding the effect of removing a vertex from a graph on this number.

Keywords: Transversal, Colour Transversal, Vertex Covering Set, Vertext Covering Number, Complementary Colour Transversal Vertex Covering Set, Complementary Colouring, Complementary Chromatic Number, Complementary Colour Transversal Vertex Covering Number.

AMS Subject Classification (2010): 05C15, 05C69.

1. INTRODUCTION

The concept of a vertex covering set is well known and has been studied by several authors. The identity $\alpha_0(G) + \beta_0(G) = |V(G)|$ ($\alpha_0(G)$ = The vertex covering number & $\beta_0(G)$ = The independence number) is well known. The concept of colour transversal dominating set was studied in detail in Ph.D. Thesis of Manoharan [9]. We introduce the concepts of colour transversal vertex covering set and colour transversal vertex covering number of a graph in [3].

In this paper we consider the concepts of complementary colouring and complementary chromatic number of a graph. These concepts were introduced in [2]. Now we introduce the concepts of Complementary Colour Transversal Vertex Covering Set (CCTVC Set) and Complementary Colour Transversal Vertex Covering Number (CCTVC Number) of a graph. The operation of removing a vertex from a graph may increase, decrease or keep the number unchanged. We consider the effect of this operation on complementary colour transversal vertex covering number (CCTVC Number) of a graph.

We assume that our graphs are finite, simple and undirected. If G is a graph then V(G) will denote the vertex set of G and E(G) will denote the edge set of G.

2. RESULTS AND DISCUSSION

Definition 2.1 (Complementary Colouring) [2]

Let G be a graph. The Colouring $f$ of vertices of G is said to be a complementary colouring if whenever vertices u and v have different colours then they must be adjacent.

Definition 2.2 (Complementary Chromatic Number) [2]

Let G be a graph. The maximum numbers of colours which can be assigned to the vertices so that the resulting colouring is a complementary colouring is called the complementary chromatic number of G & it is denoted as $\chi_c(G)$. This complementary colouring is called complementary chromatic colouring.

Remark 2.3

- The complementary colouring of a graph need not be a proper colouring.
- If a graph G has having complementary colouring then it may happen that two vertices are adjacent and they have the same colour.
If a graph has been given a complementary colouring then two non-adjacent vertices cannot have different colours. Thus, in any independent set all the vertices must have the same colours.

It may be noted that in general a colour class corresponding to a complementary colouring need not be an independent set.

**Example 2.4**

Consider the graph with vertices \( v_1, v_2, v_3, v_4 \)

Consider complementary colouring in which \( v_1, v_2, v_3, v_4 \) receives colours as follows.

\( v_1 \) – colour 1, \( v_2 \) – colour 1, \( v_3 \) – colour 2, \( v_4 \) – colour 1

Here the colour classes corresponding to colour 1 is not an independent set.

**Proposition 2.5** [2]

Let \( G \) be a graph. Then

- \( \chi_C(G) \leq \chi(G) \)
- \( \chi_C(G) = \chi(G) \) iff \( G \) is a complete \( k \)–partite graph.

**Proposition 2.6**

Let \( G \) be a graph and suppose the colour classes of a complementary chromatic colouring of \( G \) are \( C_1, C_2, \ldots, C_k \). Let \( T \) be a transversal of these colour classes then \( T \) is a dominating set.

**Proof**

Let us assume that \( T \) intersect each \( C_i \) in a singleton set and therefore let \( T \cap C_i = \{ v_i \} \) for \( i = 1, 2, \ldots, k \). Let \( z \) be a vertex such that \( z \) does not belongs to \( T \). Suppose \( z \in C_i \) for some \( i \). Then \( z \) is adjacent to \( v_i \) for every \( j \neq i \).

Thus, \( T \) is a dominating set.

**Corollary 2.7**

Let \( G \) be a graph. Then \( \gamma(G) \leq \chi_c(G) \)

**Proof**

From the above proposition \( \gamma(G) \leq |T| = \chi_c(G) \)

**Proposition 2.8**

Let \( G \) be a graph and \( C_1, C_2, \ldots, C_k \) be the colour classes corresponding to some complementary chromatic colouring of \( G \). Then for every colour class \( C_i \) with \( |C_i| \geq 2 \) & for every \( v \in C_i \exists \) some \( u \in C \) \( u \) is not adjacent to \( v \).

**Proof**

Suppose the statement does not hold.

Then for some colour class say \( C_1 \) with \( |C_1| \geq 2 \) there is a vertex \( v \) in \( C_1 \) such that \( v \) is adjacent to every vertex of \( C_1 \). Also \( v \) is adjacent to every vertex of every other colour class. Thus \( v \) is adjacent to every other vertex of \( G \). Now, suppose we have used colours 1, 2, 3, \ldots, \( k \) in complementary
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chromatic colouring of G. We may assign a new colour \( k + 1 \) to \( v \) and keep the colours of other vertices unchanged. Then we get a complementary colouring of G with \( k + 1 \) colours. This is a contradiction because complementary chromatic number of G = k.

Therefore the statement of the proposition must be true.

**Proposition 2.9**

Let G be a graph and suppose \( C_1, C_2, \ldots, C_k \) are the colour classes corresponding to some complementary colouring of G. Let T be an independent subset of G. Then \( T \subseteq C_i \) for some i.

**Proof**

If T is a singleton set then obviously \( T \subseteq C_i \) for some i.

Suppose T has at least two elements and suppose \( T \cap C_i \neq \phi \) and \( T \cap C_j \neq \phi \) for some \( i \neq j \).

Let \( v \in T \cap C_i \) and \( u \in T \cap C_j \). Since \( v \in C_i \) and \( u \in C_j \) and \( i \neq j \) v and u must be adjacent. This contradicts the fact that T is an independent set.

\[ \therefore \ T \text{ cannot intersect two distinct colour classes. Also } T \cap C_i \text{ is non-empty because the colour classes forms a partition of } V(G). \text{ Thus } T \subseteq C_i \text{ for some } i. \]

The following theorem is proved in [1]. We present a different proof for the sake of completeness.

**Theorem 2.10**

Let G be a graph then the complementary chromatic colouring of G is unique. (in the sense that any two complementary chromatic colouring of G give rise to the same colour classes)

**Proof**

Suppose there are two complementary chromatic colouring of G whose colour classes are \{ \( C_1, C_2, \ldots, C_k \) \} and \{ \( D_1, D_2, \ldots, D_k \) \}. We will prove that for every \( i \) \( C_i = D_j \) for some unique \( j \).

For this first we prove that for every \( i \) there is some \( j \geq C_i \subseteq D_j \)

Since \( C_i \neq \phi \) \& \( D_1 \cup D_2 \cup \ldots \cup D_k = V(G), C_i \cap D_j \neq \phi \) for some \( j \)

**Claim**

\( C_i \subseteq D_j \)

**Proof**

Suppose \( C_i \cap D_j \neq \phi \) for some \( j \) \& for some \( j' \) \( C_i \cap D_{j'} \neq \phi \). For the sake of simplicity we assume that \( C_i \) intersects only these two sets \( D_j \) \& \( D_{j'} \).

Let \( C_{i'} = C_i \cap D_j \) \& \( C_{i'} = C_i \cap D_{j'} \)

\[ \therefore C_{i'} \cup C_{i'} = C_i \]

Now we assign a new colouring to vertices of G as follows.

For every \( r \neq i \) the colours of vertices of the colour class \( C_r \) are unchanged.

If \( x \in C_i \cap D_j \) then we assign colour \( i' \) to \( x \).

If \( x \in C_i \cap D_{j'} \) then we assign colour \( i'' \) to \( x \).

Then we have a new complementary chromatic colouring of G consisting of colours \( 1, 2, 3, \ldots, i - 1, i', i'', i + 1, \ldots, k \).

This colouring uses \( k + 1 \) colours \& it is a complementary colouring. This contradicts the fact that the complementary chromatic number of G is k.

\[ \therefore C_i \cap D_j \neq \phi \text{ for unique } j. \]

\[ \therefore C_i \subseteq D_j \text{ for some unique } j. \]
If $C_i$ is a proper subset of $D_j$ for some $i$ then $C_1 \cup C_2 \cup \ldots \cup C_k \neq V(G)$ because $D_1 \cup D_2 \cup \ldots \cup D_k = V(G)$.

Thus $C_i = D_j$ for some unique $j$.

$\therefore \{C_1, C_2, \ldots, C_k\} = \{D_1, D_2, \ldots, D_k\}$.

This proves that this colouring is unique.

**Proposition 2.11**

Let $G$ be a graph and $v \in V(G)$. Let $f$ be a complementary colouring of $G$ then the restriction $g$ of $f$ on $G - v$ is also a complementary colouring of $G - v$.

**Proof**

Let $x$ and $y$ be two vertices of $G - v$ such that $g(x) \neq g(y)$ then $f(x) \neq f(y)$.

Since $f$ is a complementary colouring, it follows that $x$ and $y$ are adjacent vertices of $G$ and therefore adjacent vertices of $G - v$.

**Theorem 2.12**

Let $G$ be a graph and $v \in V(G)$. Then the following statements are equivalent

1. $\chi_c(G - v) < \chi_c(G)$
2. $v$ is adjacent to every other vertex of $G$.
3. $\{v\}$ is colour class in the complementary chromatic colouring of $G$.

**Proof**

$(1) \Rightarrow (3)$

Suppose $\{v\}$ is not a colour class in the complementary chromatic colouring of $G$. Therefore there is a vertex different from $v$ which has the same colour as $v$. Now, consider the restriction $g$ of the complementary chromatic colouring $f$ of $G$. There is a vertex $u$ in $G - v$ such that $f(u) = f(v)$. Then $g$ is a complementary chromatic colouring of $G - v$. Also $g$ is a complementary colouring of $G - v$.

$\therefore \chi_c(G - v) \geq$ The number of colours used by $g = \text{The number of colours used by } f = \chi_c(G)$

$\therefore \chi_c(G - v) \geq \chi_c(G)$

This is a contradiction.

$\therefore \{v\}$ is colour class in the complementary chromatic colouring of $G$.

$(3) \Rightarrow (2)$

For any complementary colouring of a graph $G$ a vertex in any colour class is adjacent to every vertex in every other colour class. Since $\{v\}$ is a colour class, $\{v\}$ is adjacent to every vertex of every other colour class. Equivalently $v$ is adjacent to every other vertex of $G$.

Therefore $(2)$ is proved.

$(2) \Rightarrow (1)$

Suppose $v$ is adjacent to every other vertex of $G$.

Consider any complementary chromatic colouring of $G - v$ which uses colours $1, 2, 3, \ldots, k$. Now, assign colour $k + 1$ to $v$. Then obviously we get a complementary colouring of vertices of $G$ which uses $k + 1$ colours.

$\therefore \chi_c(G) \geq k + 1 > k = \chi_c(G - v)$

$\therefore \chi_c(G - v) < \chi_c(G)$

**Corollary 2.13**

Let $G$ be a graph and $v \in V(G)$. If $\chi_c(G - v) = \chi_c(G)$ then $\{v\}$ is not a colour class in the complementary chromatic colouring of $G$. 
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**PROOF**

Since \( \chi_c(G - v) = \chi_c(G) \)
\( \chi_c(G - v) \not\subset \chi_c(G) \)

So, \( \{v\} \) is not a colour class in the complementary chromatic colouring of \( G \).

**Definition 2.14 (Complementary Colour Transversal Vertex Covering Set)**

Let \( G \) be a graph. A subset \( S \) of \( V(G) \) is said to be a complementary colour transversal vertex covering set of \( G \) if

1. \( S \) is a transversal for the complementary chromatic colouring of \( G \) and
2. \( S \) is a vertex covering set of \( G \)

This set is also called CCTVC set of \( G \).

**Example 2.15**

For the graph mentioned in example – 2.4, \( S = \{v_1, v_3\} \) is a CCTVC set.

**Definition 2.16 (Complementary Colour Transversal Vertex Covering Number)**

Let \( G \) be a graph and \( S \subseteq V(G) \). If \( S \) is a complementary colour transversal vertex covering set of \( G \) whose cardinality is minimum among all complementary colour transversal vertex covering set of \( G \) then \( S \) is said to be a minimum complementary colour transversal vertex covering set of \( G \).

The cardinality of such a set is called complementary colour transversal vertex covering number (or CCTVC Number) of \( G \). It is denoted as \( \alpha_{c,c}(G) \).

**Theorem 2.17**

Let \( G \) be a graph. Then for \( G \) only one of the following two possibilities holds.

1. \( \alpha_{c,c}(G) = \alpha_0(G) \)
2. \( \alpha_{c,c}(G) = \alpha_0(G) + 1 \)

**Proof**

Let \( G \) be a graph. Consider any complementary chromatic colouring of \( G \) and suppose \( C_1, C_2, \ldots, C_k \) are the colour classes corresponding to this colouring. Let \( S \) be a maximum independent subset of \( G \) so that \( |S| = \beta_0(G) \). Now \( S \) is a subset of \( C_i \) for some unique \( i \). Suppose \( S \) is a proper subset of \( C_i \) then,

1. \( V(G) - S \) is a minimum vertex covering set of \( G \).
2. \( V(G) - S \) is a colour transversal for this complementary colouring of \( G \)

Therefore, \( V(G) - S \) is a minimum vertex covering set as well as a complementary colour transversal vertex covering set.

Since \( \alpha_{c}(G) \geq \alpha_0(G) \) it follows that \( \alpha_{c,c}(G) = \alpha_0(G) \) in this case.

Suppose \( S \) is a subset of \( C_i \) and \( S = C_i \) then \( V(G) - S \) is a vertex covering set but it is not a transversal for this colouring. Let \( x \) be any vertex of \( S \) then the set \( (V(G) - S) \cup \{x\} \) is a CCTVC set of \( G \).

Let \( T = (V(G) - S) \cup \{x\} \)
\( \therefore \alpha_{c,c}(G) = |T| = |V(G) - S| + 1 = \alpha_0(G) + 1 \)

Thus for any graph \( G \) only one of the following two possibilities holds

1. \( \alpha_{c,c}(G) = \alpha_0(G) \)
2. \( \alpha_{c,c}(G) = \alpha_0(G) + 1 \)
Theorem 2.18

If G is a complete graph then for any vertex v of G

1. \( \chi_C(G - v) < \chi_C(G) \)
2. \( \alpha_c(G - v) < \alpha_c(G) \)

Proof

Result (1) follows from Theorem 2.17

(2) Suppose \(|V(G)| = n\). Since G is a complete graph \( \chi_C(G) = n \) and \( \alpha_c(G) = n \) for any \( v \in V(G) \), G – v is also a complete graph.

\[ \therefore \alpha_c(G - v) = n - 1 < n = \alpha_c(G) \]

\[ \therefore \alpha_c(G - v) < \alpha_c(G) \]

Theorem 2.19

Let G be a graph with \( \beta_0(G) \geq 2 \). Let \( v \in V(G) \) \( \chi_C(G - v) < \chi_C(G) \) then \( \alpha_c(G - v) < \alpha_c(G) \)

Proof

Since \( \beta_0(G) \geq 2 \), G is not a complete graph. First suppose that \( \alpha_c(G) = \alpha_0(G) \).

Let S be a minimum vertex covering set of G. Now, \( V(G) - S \) is a maximum independent set of G.

\[ \therefore v \not\in V(G) - S \quad (\because v \text{ is adjacent to every other vertex of } G \& \beta_0(G) \geq 2) \text{ and therefore } v \in S. \]

Now, \( S_1 = S - \{v\} \) is a vertex covering set of \( G - v \) also \( S_1 \) is a colour transversal for the complementary chromatic colouring of \( G - v \) which is induced from the complementary chromatic colouring of G.

\[ \therefore S_1 \text{ is a CCTVC set of } G - v. \]

\[ \therefore \alpha_c(G - v) = |S_1| < |S| = \alpha_c(G) \]

Suppose \( \alpha_c(G) = \alpha_0(G) + 1 \)

Let S be a minimum CCTVC set of G then \( v \in S \) because \( \{v\} \) is a colour class in the unique complementary chromatic colouring of G.

Now, let \( S_1 = S - \{v\} \) then \( S_1 \) is a CCTVC set of \( G - v \).

\[ \therefore \alpha_c(G - v) = |S_1| = \alpha_0(G) < \alpha_c(G) \]

\[ \therefore \alpha_c(G - v) < \alpha_c(G) \]

Theorem 2.20

Let G be a graph \& \( v \in V(G) \). If \( \chi_C(G - v) = \chi_C(G) \) then \( \alpha_c(G - v) \leq \alpha_c(G) \)

Proof

Since \( \chi_C(G - v) = \chi_C(G) \), \( \{v\} \) is not a colour class in the complementary chromatic colouring of G.

Let S be a minimum CCTVC set of G.

Case 1: \( v \notin S \)

Then S is a vertex covering set of \( G - v \) & since it is a colour transversal of G it contains a vertex u different from v such that u has the same colour as v.

Thus S is a CCTVC set in \( G - v \).

Case 2: \( v \in S \)

Suppose S contains a vertex u different from v which has the same colour as v. Then \( S - \{v\} \) is a vertex covering set of \( G - v \) and it is also a colour transversal for the complementary chromatic colouring of \( G - v \).
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Suppose \( v \in S \) & there is no other vertex which has the same colour as \( v \) & which is in \( S \).

In this case let \( u \) be a vertex different from \( v \) such that \( u \) has the same colour as \( v \).

Let \( S_1 = (S - \{v\}) \cup \{u\} \)

Then \( S_1 \) is a CCTVC set.

From both the cases above it follows that \( \alpha_c(G - v) \leq \alpha_c(G) \)

Now, we consider the possibility when \( \chi_c(G - v) > \chi_c(G) \).

Let \( \{C_1, C_2, \ldots, C_j\} \) be the set of all colour classes of \( G \) (\( j \geq 1 \)) and let \( \{D_1, D_2, \ldots, D_k\} \) be the set of all colour classes of \( G - v \).

**Theorem 2.21**

\( \chi_c(G - v) > \chi_c(G) \) iff

(1) There are at least two colour classes of \( G - v \) which are all subsets of the colour class \( C \) which contains \( v \) & there union = \( C - \{v\} \) and \( v \) is non-adjacent with some vertex in every such colour class.

(2) Other colour classes of \( G - v \) are just the colour classes of \( G \) different from \( C \).

**Proof**

(1) Suppose \( \chi_c(G - v) > \chi_c(G) \) then \( k > j \)

Now, each colour class \( D_i \) intersect some colour class \( C_r \) of \( G \). Suppose \( D_i \cap C_r \neq \emptyset \) & \( D_i \cap C_r' \neq \emptyset \).

Now, let \( D_i \cap C_r = D_i' \) & \( D_i \cap C_r' = D_i'' \).

Then we can assign two distinct colours of vertices of \( D_i' & D_i'' \) in place of the single colour of \( D_i \).

This will increase the number of colour used in complementary colouring of \( G - v \). Which is a contradiction.

\[ \therefore D_i \cap C_r \neq \emptyset \text{ for some unique } r. \]

\[ \therefore D_i \subseteq C_r \text{ for some unique } r. \]

Also, there are colour classes of \( G - v \) which intersect the colour class \( C \) containing \( v \). Therefore, there are colour classes which are subsets of \( C \). Suppose there are only \( m \) colour classes of \( G - v \) which are containing in \( C \) and \( m < k - j + 1 \).

Now, we provide a new colouring of \( G \) as follows.

Assign the same colour as that of \( C \) to all the vertices which belong to the \( m \) colour classes mentioning above. Do not change the colours of the remaining \( j - m \) colour classes which are disjoint from \( C \). Thus we get a complementary chromatic colouring of \( G \) consisting of \( k - m + 1 \) colours, which is greater than \( j \). This is a contradiction as \( j \) is the highest number of colours which is assign to vertices of \( G \) so that resulting colouring is complementary colouring.

Suppose there are \( m \) colour classes of \( G - v \) which are containing in \( C \) and \( m > k - j + 1 \).

Then \( k - m < j - 1 \)

Thus it must be true that the remaining \( k - m \) colour classes of \( G - v \) are contained in \( j - 1 \) colour classes of \( G \). Which is impossible because \( k - m < j - 1 \).

Thus, \( m < k - j + 1 \) & \( m > k - j + 1 \) are impossibilities. Therefore, \( m = k - j + 1 \)

Since \( C \) is a colour class in \( G \) containing \( v \) & union of the above mentioned colour classes = \( C - \{v\} \), \( v \) must be non-adjacent to some vertex in the union & therefore \( v \) must be non-adjacent to some vertex in some colour class.

**Claim**

Now, we prove that \( v \) is non-adjacent with some vertex in every colour class of \( G - v \) which is contained in \( C \).
**PROOF OF THE CLAIM**

Suppose there is a colour class of \( G - v \) say \( D \) such that \( D \subseteq \mathcal{C} \) & \( v \) is adjacent with every vertex of \( D \). Then we can assign a colour to the vertices of \( D \) which is different from \( v \) & it is also different from the colours of other colour classes of \( G \).

Thus we get a complementary colouring of \( G \) which consists of \( j + 1 \) colours. This contradicts the fact that complementary chromatic number of \( G = j \). Therefore, there is no colour class of \( G - v \) which is contained in \( D \) & \( v \) is adjacent with every vertex of that colour class.

(2) Now, consider the remaining \( k - (k - j + 1) = j - 1 \) colour classes of \( G - v \). Since there are \( j - 1 \) colour classes of \( G \) different from \( C \), each colour class is contained in a unique colour class of \( G \). Since the union of both the colour classes = \( V(G) \), this \( j - 1 \) colour classes of \( G - v \) are exactly the colour classes different from \( C \).

Conversely suppose (1) and (2) holds then it follows that

The number of colour classes of \( G - v \) > The number of colour classes of \( G \)

\[ \therefore \chi_C(G - v) > \chi_C(G) \]

**THEOREM 2.22**

Let \( G \) be a graph & \( v \in V(G) \). Suppose \( \chi_C(G - v) > \chi_C(G) \) than any of the following three possibilities can hold

(1) \( \alpha_c(G - v) > \alpha_c(G) \)
(2) \( \alpha_c(G - v) = \alpha_c(G) \)
(3) \( \alpha_c(G - v) < \alpha_c(G) \)

**PROOF**

First suppose that \( \alpha_0(G - v) < \alpha_0(G) \) then there is a minimum vertex covering set \( S \) of \( G \) such that \( v \in S \). Let \( M = V(G) - S \) then \( M \) is a maximum independent subset of \( G \) & \( v \not\in M \).

Now, \( M \subseteq C_i \) for some \( i \)

Case (1) \( M \) is a proper subset of \( C_i \)

Then \( V(G) - M = S \) is a minimum vertex covering set & it is also a colour transversal of \( G \).

\[ \therefore S \text{ is a CCTVC set of } G \& |S| = n - \beta_0(G) = \alpha_0(G) \]

Since \( M \) does not contain \( v \), \( M \) is also a maximum independent subset of \( G - v \). Therefore \( M \) is a subset of \( D_i \) for some unique \( r \).

If \( M \) is a proper subset of \( D_i \) then Let \( G_i = G - v \)

\[ \therefore S_i = V(G_i) - M \text{ is a minimum vertex covering set of } G - v \& \text{ it is also a colour transversal of } G - v \]

\[ \therefore S_i \text{ is a CCTVC set of } G - v \& |S_i| = n - \beta_0(G) \]

\[ \therefore \alpha_c(G - v) < \alpha_c(G) \]

Now, suppose \( M = D_i \). Let \( x \in D_i \)

Consider the set \( S_1 = (V(G_i) - M) \cup \{x\} \) then \( S_1 \) is a vertex covering set & it is also a colour transversal of \( G - v \).

Also \( |S_1| = \alpha_0(G) + 1 \)

\[ \therefore S_1 \text{ is a CCTVC set of } G - v \]

\[ \therefore \alpha_c(G - v) = |S_1| = n - \beta_0(G) = \alpha_c(G) \]

Case (2) Suppose \( M = C_i \) for some \( i \). Let \( x \in M \).
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Now, consider the set \( T = ( V(G) - M ) \cup \{ x \} \) then \( T \) is a vertex covering set of \( G \) & it is also a colour transversal for complementary chromatic colouring of \( G \). Thus \( T \) is a minimum CCTVC set.

\[
\therefore \alpha_c(G) = | T | = n - \beta_0(G) + 1
\]

Suppose \( M \) is a proper subset of some colour class \( D_i \) of the complementary chromatic colouring of \( G - v \). Then \( T_1 = V(G_1) - M \) is is a minimum vertex covering set of \( G - v \) & it is also a colour transversal for this colouring of \( G - v \).

\[
\therefore \alpha_c(G - v) = | T_1 | = n - 1 - \beta_0(G)
\]

\[
\therefore \alpha_c(G - v) < \alpha_c(G)
\]

On the other hand if \( M = D_i \) for some \( r \) then let \( y \in D_i \)

Then \( T_2 = ( V(G_1) - M ) \cup \{ y \} \) is a vertex covering set & it is also a colour transversal of \( G - v \).

\[
\therefore \alpha_c(G - v) = n - 1 - \beta_0(G)
\]

\[
\therefore \alpha_c(G - v) < \alpha_c(G)
\]

Now suppose \( \alpha_0 (G - v) = \alpha_0 (G) \)

In this case \( v \notin S \) for any minimum vertex covering set \( S \) of \( G \).

\[
\therefore v \in M \text{ for every maximum independent subset } M \text{ of } G. \text{ Now, } M \text{ is a subset of } C_i \text{ for some colour class } C_i. \text{ Since } v \in M \Rightarrow v \in C_i
\]

Suppose \( M \) is a proper subset of \( C_i \) then as proved above \( \alpha_c(G) = n - \beta_0(G) \)

1. Suppose \( M - \{ v \} \) is a proper subset of \( D_i \) for some colour class \( D_i \) of \( G - v \).

Then \( \alpha_c(G - v) = n - 1 - ( \beta_0(G) - 1 ) = n - \beta_0(G) \)

\[
\therefore \alpha_c(G - v) = \alpha_c(G)
\]

2. Suppose \( M - \{ v \} = D_i \) for some colour class \( D_i \) of \( G - v \).

Then \( \alpha_c(G - v) = n - 1 - ( \beta_0(G) - 2 ) = n - \beta_0(G) + 1 \)

\[
\therefore \alpha_c(G - v) > \alpha_c(G)
\]

Suppose \( M = C_i \) for some colour class \( C_i \) of \( G \).

Then \( \alpha_c(G) = n - \beta_0(G) + 1 \)

Suppose \( M - \{ v \} \) is a proper subset of \( D_i \) for some colour class \( D_i \) of \( G - v \). As proved in above theorem \( D_i \) & \( D_s \) are subsets of \( C_i \) for at least two distinct values \( r \) & \( s \) then \( M - \{ v \} \) will be a proper subset of \( D_i \cup D_s \), and therefore \( M - \{ v \} \) will be proper subset of \( C - \{ v \} \).

\[
\therefore M \text{ is proper subset of } C_i \text{ which is contradiction.}
\]

\[
\therefore M \text{ is proper subset of } D_i \text{ is not possible for any } r.
\]

Hence, \( \alpha_c(G - v) = n - 1 - ( \beta_0(G) - 2 ) = n - \beta_0(G) + 1 \)

\[
\therefore \alpha_c(G - v) = \alpha_c(G)
\]

**Example 2.23**

Consider the graph \( G \) in example 2.4

Here, \( \chi_C (G) = 2 \) & \( \chi_C (G - v_4) = 3 \)

\[
\therefore \chi_C (G - v) > \chi_C (G)
\]

Also observe that

\[
\alpha_c(G) = 2 \text{ & } \alpha_c(G - v_4) = 3
\]

Hence, \( \alpha_c(G - v) > \alpha_c(G) \)
Example 2.24
Consider the path graph with four vertices \( G = P_4 \)

\[ \chi_c(G) = 1 \quad \text{&} \quad \chi_c(G - v_4) = 2 \]

\[ \therefore \chi_c(G - v) > \chi_c(G) \]

Also observe that
\[ \alpha_c(G) = 2 = \alpha_c(G - v_4) \]

Hence, \( \alpha_c(G - v) = \alpha_c(G) \)

3. CONCLUDING REMARK

There are enough number of examples of graph \( G \) for which \( \chi_c(G - v) > \chi_c(G) \) and \( \alpha_c(G - v) \geq \alpha_c(G) \). However, we do not know a graph \( G \) for which \( \chi_c(G - v) > \chi_c(G) \) and \( \alpha_c(G - v) < \alpha_c(G) \).

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AUTHORS’ BIOGRAPHY

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