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Note on the Vector Space $\mathcal{B}(H)$ of Bounded Operators an a Separable Hilbert's space H

Joseph S.V. Massamba

Département de Mathématique et Informatique, Faculté des Sciences
Université Pédagogique Nationale (UPN)
Kinshasa, RD. Congo

Abstract: *This paper aims at solving the problem of knowing whether we can find the vector space $\mathcal{B}(H)$ of bounded operators on a separable Hibert's space H and a scalar product and eventually decide on the completeness of hermitian norm. I did not only succeed to confer the structure of Hilbert's space to the vector space $\mathcal{B}(H)$, but also to establish equality between norm operator and hermitian norm on $\mathcal{B}(H)$.*

Keywords: *Vector space, scalar product, completeness, separable, Hilbert space, Banach space, norm operator, Hilbert basis*

USEFUL MATTERS

1.1 Scalar Product

Let E be a vector space and a map denoted $g : ExE \to \mathbb{K}(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$ filling following properties :

$$(i)\ \forall\ x, y, z \in E: g(x + y, z) = g(x, z) + g(y, z)$$

$$(ii) \forall\ x, y, z \in E: g(x, y + z) = g(x, y) + g(x, z)$$

$$(iii) \forall\ x, y \in E \text{ and } r \in \mathbb{K}: \ g(rx, y) = r(x, y)$$

$$(iv) \forall\ x, y \in E \text{ and } r \in \mathbb{K}: g(x, ry) = \bar{r}(x, y)$$

$(v) \forall\ x, y \in E : g(x, y) = \overline{g(y, x)}$

$(iv) \forall\ x, \in E :\ g(x, x) > 0 \text{if} x \neq 0 \text{and} g(x, x) = 0 \text{if}\ x = 0$

The so-defined map g is called hermitian form or simply a scalar product on E. Note that if $\mathbb{K} = \mathbb{R}$ then g is a bilinear form and, consequently properties (ii) and $(v)$are dropped [1, 2, 3].

1.2 Two Notions Generated by the Scalar Product

1.2.1. First Notion

The map denoted

$$\|\quad\| : E \to \mathbb{R}$$

$$x \mapsto \|x\| = \sqrt{< x, x >}$$

With properties :

$(i) \forall\ x \in E,\ \|x\| > 0 \text{if} x \neq 0 \text{ and} \|x\| = 0 \text{if} x = 0$

$(ii) \forall\ x \in E$andascalar r : $\|rx\| = r\|x\|$

$(iii) \forall\ x, y \in E : \|x + y\| \leq \|x\| + \|y\|$

This mapis called hermitian norm on E

1.2.1. Second Notion

Inequality $|(x,y)| \leq \|x\|\|y\| \; \forall \, x,y \in E$, is called Cauchy-Schwarz's inequality.

1.3 Hilbert's Space

1.3.1 Definition

Let E be a vector space provided with a scalar product; it is said that E is a space of Hilbert if the associated hermitian norm is complete; in other words, if any Cauchy's sequence in E is convergent; a space of Hilbert E is known as separable if it possesses a dense and countable part or simply a hilbertian basis.

1.3.2 Remark

In this note, H is aseparable complex Hilbert's space with infinite size whose scalar product, norm and hilbertian basisare respectively denoted $(\,,), \|\quad\|$ and$b = (e_i)_{i\geq 1}$; its null element 0_H is simply denoted 0; the bounded operators, also called continuous operators on H, are appointed by capital letters A, B, C..; their set is denoted $\mathcal{B}(H)$; $B_H = \{x \in H : \|x\| \leq 1\}$ is the closed unit bowlof H; any vector x of H is represented by $x = \sum_{i=1}^{\infty} u_i e_i$ and $\sum_{i=1}^{\infty} |u_i|^2 = \|x\|^2$ with $u = (x, e_i)$ (1); for any operator A and $e_i \in b$, one will write Ae_i instead of$A(e_i)$, the field of definition of an operator A on H is dense in H, which means that $\bar{A} = H$, $\bar{A}$ being the adherence or the closing of A; thus, for both bounded operators A and B on H and $e_{\hat{\imath}} \in b, Ae_i$ and Be_i are vectors of H such as $(Ae_i \,, Be_i) \in \mathbb{K} = (\mathbb{R} \; ou \; \mathbb{C})[2]$; the norm operator on H is, in general denoted and defined by$\|A\|_1 = sup\{\|Ax\| : x \in$ BH or, in particular$A1 = supAei:i \in b$.

1.3.3 Proposition

$\forall$ A a bounded operator on H anda vector$x \in B_H$, onehas :$\|Ax\|^2 \leq (\sum_{i=1}^{\infty} |u_i|\|Ae_i\|)^2$

Proof

Onehas $\|Ax\|^2 = \|A(\sum_{i=1}^{\infty} u_i e_i)\|^2$

$$= \left\|\sum_{i=1}^{\infty} A(u_i, e_i)\right\|^2 = \left\|\sum_{i=1}^{\infty\sqrt{}} u_i \, Ae_i\right\|^2$$

$= (\sum_{i=1}^{\infty} u_i Ae_i \,, \sum_{i=1}^{\infty} u_i Ae_i)$ [scalar product on H]

$$\leq \left|\left(\sum_{i=1}^{\infty} u_i Ae_i \,, \sum_{i=1}^{\infty} u_i Ae_i\right)\right|$$

$\leq \|\sum_{i=1}^{\infty} u_i Ae_i\|\|\sum_{i=1}^{\infty} u_i Ae_i\|$ [inequality of Cauchy-Schwarz]

$= \sum_{i=1}^{\infty} \|u_i Ae_i\| \sum_{i=1}^{\infty} \|u_i Ae_i\|$[continuity of the norm]

$$= \sum_{i=1}^{\infty} |u_i|\|u_i Ae_i\| \sum_{i=1}^{\infty} |u_i|\|Ae_i\| = \left(\sum_{i=1}^{\infty} |u_i|\|Ae_i\|\right)^2$$

Hence one obtains$\|Ax\|^2 \leq (\sum_{i=1}^{\infty} |u_i|\|Ae_i\|)^2$ [1, 2, 3]

SEEKING EFFICIENT SCALAR PRODUCT

1.4 Method

Let H be a separable complex space of Hilbert with infinite size; one resorts to the convergence of sequences in normalized spaces, in fact, spaces of Banach $\mathbb{K} = (\mathbb{R} \text{ or } \mathbb{C})[\;1\;]$ and, also with the exploitation of elements presented above such as if A and B are two bounded operators on H and

$e_i \in b$ then the fields of definition of A and B are dense, $(Ae_i, Be_i) \in \mathbb{C}$ and $|Ae_i, Be_i| \leq \|Ae_i\|\|Be_i\|$ the inequality of Cauchy-Schwarz.

1.5 Establishment

Let A and B be two bounded operators on H and$e_i \in b$ then; $(Ae_i, Be_i) \in \mathbb{C}$; one considers the numerical series $\sum_{i=1}^{\infty} \frac{1}{2^i} |(Ae_i, Be_i)|$;it is clear that one has successively

$\sum_{i=1}^{\infty} \frac{1}{2^i} |(Ae_i, Be_i)| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|\|Be_i\|$ [inequality of Cauchy-Schwarz]

$\leq \|A\|_1 \|B\|_1 < \infty$ [for$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$]

or simply the inequality $\sum_{i=1}^{\infty} \frac{1}{2^i} |(Ae_i, Be_i)| < \infty$ which means that the obtained series $\sum_{i=1}^{\infty} \frac{1}{2^i} |(Ae_i, Be_i)|$converges in $\mathbb{R}^+$; then it results from it that the series $\sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i)$converge absolutely in $\mathbb{R}$ and, consequently, it converges in$\mathbb{C}$; that is to say that $\sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i) \in \mathbb{C}$[3] such as, for any bounded operator A on H, $\sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Ae_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|^2 \geq 0$ which means that $\sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|^2 > 0$ for all $A \neq 0$ and $\sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|^2 = 0$ when$A = 0$; the scalar obtained is denoted $< A, B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i)$ for any bounded operator A on H; now it is shown that it is independent of the choice of the used hilbertian basis; indeed, if $d = (g_t)_{t \geq 1}$another hilbertian basis of H then one obtains successively

$$(Ae_i, Be_i) = (Ae_i, g_t)(g_t, Be_i) = (e_i, A * g_t)(B * g_t, e_i)$$
$$= (B * g_t, A * g_t) = (Ag_t, Bg_t)$$

It results from it that one obtains the equality$\sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} (Ag_t, Bg_t)$; however, that is not enough to conclude that $< , >_2$ is a scalar product on H; it should be shown that $< , >_2$ enjoys the properties(1.1).

1.6 Theorem

The map$< , >_2 : H \times H \to \mathbb{C}$enjoys the properties: (1.1)Indeed, one has respectively

(i) For all three bounded operators A, B, C on H and: $e_i \in \varphi$

$$< A + B, C >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} [(Ae_i, Ce_i) + (Be_i, Ce_i)] = < A, C >_2 + < B, C >_2$$

(ii) Whatever two bounded operators A, B on H and: $e_i \in \varphi$

$$< A, B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i) = \overline{\sum_{i=1}^{\infty} \frac{1}{\acute{e}^i} (Be_i, Ae_i)} = \overline{< B, A >_2}$$

(iii) For both bounded operators A, B on H , λ a scalar and: $e_i \in \varphi$

$< \lambda A, B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (\lambda Ae_i, Be_i) = \lambda \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i)$ while

$$< A, \lambda B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (\lambda Ae_i, Be_i) = \bar{\lambda} \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i)$$

(iv) Whatever abounded operator A on H and $e_i \in \varphi$ one has on the one hand, $< A, A >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Ae_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \|\mathrm{Ae_i}\|^2 \geq 0$ with $i = 1,2,3,....$, $e_i \in \varphi$ and, on the other hand < *A*,*A*>2=0⇒ *i*=1∞12*i*Aei2=0 for all *i*=1,2,3,.... and; *ei*∈*φ*that means that; *A*=0thus it is concluded-T that

$<>_2$ is a scalar product on the vector space$\mathcal{B}(H)$; the proof is finished; now one can affirm that:

RESULTS

1.7 Theorem

Let H be a separable complex space of Hilbert with infinite size whose scalar product is denoted $(,)$, $\varphi = (e_i)_{i\geq 1}$one of its hilbertian basis and the map defined by

$$<,>_2 : \left(\mathcal{B}(H)\right)^2 \longrightarrow \mathbb{C} : (A,B) \longmapsto < A,B >_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Be_i)$$

whatever two bounded operators A, B on H and $e_i \in \varphi$ then $<,>_2$is a scalar product on $\mathcal{B}(H)$; the hermitian norm associated to the scalar product is denoted and defined by $\|A\|_2 = \left(\sum_{i=1}^{\infty} \|Ae_i\|^2\right)^{1/2}$for any bounded operator A on H.

1.8 Remark

The vector space $\mathcal{B}(H)$ is thus provided with two norms, namely the norm operator $\| \quad \|_1$ and the hermitiannorm$\| \quad \|_2$;is there exist a bond between these two norms? As norms are real numbers, one resorts to the following technique to answer the asked question.

1.9 Comparing $\| \quad \|_1$ and $\| \quad \|_2$

3.3.1. Seeking the answer

It is necessary and enough to show $(k)\| \quad \|_1 \leq \| \quad \|_2$ and $(p)\| \quad \|_2 \leq \| \quad \|_1$for that, one resorts to the proposal (1.3.3)

$$(k)\| \quad \|_1 \leq \| \quad \|_2$$

Whatever a bounded operator A on H and a vector $x \in B_H$

$\|Ax\|^2 \leq \left(\sum_{i=1}^{\infty} |u_i| \|Ae_i\|\right)^2$ [proposal (1.3.3)]

$$\leq \left(\sum_{i=1}^{\infty} |u_i| \|Ae_i\|_2\right)^2 = \left(\sum_{i=1}^{\infty} |u_i| \|A\|_2 \|e_i\|\right)^2$$

[for$\|A\|_2 = \sup\{\|Ae_i\| : i = 1,2,3,\dots\}$]

$= \sum_{i=1}^{\infty} |u_i|^2 \|A\|_2^2 = \|A\|_2^2$ [for$\sum_{i=1}^{\infty} |u_i|^2 = \|e_1\|^2 = 1$]

Briefly$\|Ax\|^2 \leq \|A\|_2^2$ or simply$\|Ax\| \leq \|A\|_2$; it from of results well that $\|A\|_1 = sup\{\|Ax\| : \|x\| \leq$ 1≤𝐴2or simply 𝐴1≤𝐴2.

$$(p)\| \quad \|_2 \leq \| \quad \|_1$$

Let$A$be a bounded operator on H; then there are the following inequalities:

$\|A\|_2^2 = \sum_{i=1}^{\infty} \frac{1}{2^i} (Ae_i, Ae_i)$ [the square of the standard $\| \quad \|_2$]

$$\leq \sum_{i=1}^{\infty} \frac{1}{2^i} |(Ae_i, Ae_i)| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\| \|Ae_i\|$$

[inequality of Cauchy-Schwarz]

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} (\|Ae_i\|)^2 \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_1\|_1^2$$

[for$\|A\|_1 = sup\{\|Ae_i\| : e_i \in b\}$]

$$= \|A\|_1^2 \|e_i\|^2 \sum_{i=1}^{\infty} \frac{1}{2^i}$$

$= \|A\|_1^2$ [for$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1 = \|e_i\|^2$]

Briefly $\|A\|_2^2 \leq \|A\|_1^2$ or clearly $\|A\|_2 \leq \|A\|_1$.

3.3.2. Obtained result

The obtained results $\|A\|_1 \leq \|A\|_2$ and $\|A\|_2 \leq \|A\|_1$ mean that; $\| \quad \|_2 = \| \quad \|_1$ what we nicely express in these terms:

3.3.3. Theorem

Let H be a separable complex space of Hilbert with infinite size, $(e_i)_{i\geq 1}$ a hilbertian basis of H and the norms $\| \quad \|_1$ and $\| \quad \|_2$ on the vector space $\mathcal{B}(H)$; then the two norms are equal, in other words one has the equality $\| \quad \|_1 = \| \quad \|_2$.

1.10 Conclusion

The two theorems (3.1) and $(3.3.3)$ established clearly that, on the one hand the scalar $< A, B >_\gamma = \sum_{i=1}^{\infty} \frac{1}{2^i}(Ae_i, Be_i)$ is a square form and, on the other hand bothnorms(operator and hermitian) on the vector space $\mathcal{B}(H)$ are equal; as the norm operator $\| \quad \|_1$ is complete, it results from it that the hermitiannorm $\| \quad \|_2$ is too; that is to say that, provided with the norm $\| \quad \|_2$. the vector space $\mathcal{B}(H)$ is a space of Banach ; what leads to the short following conclusion:

Let $\mathcal{B}(H)$ be the separable complex vector space with infinite size; provided with the scalar product $<, >_\gamma = \sum_{i=1}^{\infty} \frac{1}{2^i}(Ae_i, Be_i)$, $\mathcal{B}(H)$ is a space of Hilbert.

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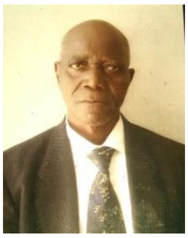
AUTHOR'S BIOGRAPHY



MasambaSalaVoka, is senior lecturer at UniversitéPédagogiqueNationale, Kinshasa (RDC). He teaches Infinitesimal Analysis, Functional Anlysis and General Topology. He is also a PhD. Student at the same University.