On Spectral Theory of Sequences Vector Spaces in a Space without Scalar Product: Case of a Banach’s Space G

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Abstract: With the question of knowing if it is possible to study sequences in a nonseparable space or not provided with a scalar product, we propose a possible technique for an unspecified Banach’s space G, with or without unit element, separable or not.

Keywords: Sequence, Convergent sequence, Vector space, Banach’s space, Hilbert’s space, Isomorphism.

1. INTRODUCTION

1.1 Useful Vector Spaces of Sequences in G

Let G be an arbitrary space of Banach, \( G^n \) the vector space of sequences in G, i.e. \( G^n = \{ x = (x_1, x_2, ..., x_i, ...) \mid x_i \in G \ for \ i = 1, 2, 3, ... \} \), \( G_C^2 = \{ x \in G^n : \exists x_\alpha \in G \ and \ \lim_{i \to \infty} x_i - x_\alpha = 0 \} \) the vectorial subspace of convergent sequences in G and, finally, the vectorial subspace of \( G_C^2 \) denoted \( G^2 \) and defined as and by the following conditions:

\[ G^2 = \{ x = (x_1, x_2, ..., x_i, ...) : x \in G_C^2 \ and \ \sum_{i=1}^{\infty} ||x_i|| < \infty \} \]; it allows two vectorial subspaces \( G_C^p \) and \( G_P^p \).

1.2 Hilbert’s spaces sequence

Now, let \( (H_i)_{i \geq 1} \) be a sequence of unspecified spaces of Hilbert and H a set of sequences defined as follows: \( H = \{ h = (h_1, h_2, ..., h_i, ...) : h_i \in H_i \ for \ i = 1, 2, 3, ... \ and \ \sum_{i=1}^{\infty} ||h_i|| < \infty \} \); it is known that H is a vector space such as, provided with the square form denoted and defined by \( \alpha : H^2 \to \mathbb{R} : \alpha(h, g) = \sum_{i=1}^{\infty} (h_i \downarrow g_i) \) for all \( h, g \in H \), it is a Hilbert’s space called hilbertian sum of the sequence \( (H_i)_{i \geq 1} \).

1.3 Towards the answer

Let us return to vector spaces \( G^2 \) and H, and consider the following maps: on the one hand \( \gamma : G^2 \to H : x \to \gamma(x) = h \in H \) for all \( x \in G^2 \) and on the other hand \( \omega : (G^2)^2 \to H^2 : (x, y) \to \omega(x, y) = (g, h) \) for all \( x, y \in G^2 \) and \( h, g \in H \); it is clear that \( \gamma \) and \( \omega \) are linear one-to-one maps, therefore isomorphisms(i); now, let \( (G^2)^2 : \varphi \to \mathbb{R} \) be a map; it is easy and favorable to note that starting from three applications \( \omega, \alpha \) and \( \varphi \), one immediately obtains the diagram presented in section 2.
2. ANSWER

2.1 Observing and exploiting the diagram

\[(G^2)^2 \varphi \rightarrow \rightarrow \mathbb{R} \]

\[\omega \quad \uparrow \]

\[\downarrow \quad \wedge \]

\[\downarrow \quad \wedge \]

\[H^2 \alpha \]

Figure 1. Commutative diagram

It is clear that \( \varphi = \alpha \omega \) such that, for all \( x, y \in G^2 \), one obtains following equalities:

\[ \varphi(x, y) = (\alpha \omega)(x, y) = \alpha(\omega(x, y)) = \alpha(g, h) = \sum_{i=1}^{\infty} (g_i \uparrow h_i) \]

or simply \( \varphi(x, y) = \sum_{i=1}^{\infty} (g_i \uparrow h_i) \)

2.2 Result

As \( H \) and \( G^2 \) are isomorphic vector spaces \((i)\) and \( H \) is a Hilbert’s space whose scalar product is \( \alpha(h, g) = \sum_{i=1}^{\infty} (h_i \uparrow g_i) \) for all \( h, g \in H \), it is easy to conclude that \( G^2 \) is a space of Hilbert whose scalar product is \( \varphi(x, y) = \sum_{i=1}^{\infty} (x_i \uparrow y_i) \) for all \( x, y \in G^2 \); thus, one can easily establish the quadruple \( G^2, G^n, G^2, G^2 \) similar to the traditional quadruple \( \mathbb{K}^n, \mathbb{K}^p, \ell^2, \ell_p \) attributed to Riesz.

2.3 An illustration

In particular, Let \( B(H) \) be a Banach’s space; by representing a sequence in \( B(H) \) by \( A = (A_1, A_2, A_3, \ldots, A_i, \ldots) \) such that \( A_i \in B(H) \) for all \( i = 1, 2, 3, \ldots \), and one can consider the vector space of the convergent sequences in \( B(H) \), denoted and defined as follows:

\[ B^2 = \{ A = (A_1, A_2, A_3, \ldots, A_i, \ldots) \text{ such that } \sum_{i=1}^{\infty} \|A_i\|^2 < \infty \} \] it is provided with two vectorial subspaces denoted \( B^p_1 \) and \( B^p_2 \); it is easy to note that the vector space \( B^2 \) is isomorphic to the vector space \( G^2 \) which is a Hilbert’s space whose scalar product is denoted \( \varphi(x, y) = \sum_{i=1}^{\infty} (x_i \uparrow y_i) \); it results from it that \( B^2 \) is a space of Hilbert whose scalar product is written as \( \delta(A, D) = \sum_{i=1}^{\infty} (A_i \uparrow D) \) for all \( A, D \in B^2 \); thus, one can easily establish the quadruple \( B_1, B_2, B_2, B_2 \) similar to the traditional quadruple \( \mathbb{K}^n, \mathbb{K}^p, \ell^2, \ell_p \) attributed to Riesz.

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REFERENCES


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