**Colour Transversal Vertex Covering Set**

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**Abstract:** In this paper we introduce new concepts called CTVC set and CTVC number of a graph. We proved that the vertex covering number of a graph is either equal to CTVC number or it is one less than the CTVC number. We have proved some results regarding the effect of removing a vertex from the graph and its effect on the CTVC number of a graph.

**Keywords:** Transversal, Colour Transversal, Vertex Covering Set, Colour Transversal Vertex Covering Set, Dominator Colouring

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1. **INTRODUCTION**

The concept of a vertex covering set is well known and has been studied by several authors. The identity $\alpha_0(G) + \beta_0(G) = |V(G)|$ (where $\alpha_0(G)$ = The vertex covering number & $\beta_0(G)$ = The independence number ) is well known. The concept of colour transversal dominating set was studied in detail in Ph.D. Thesis of Manoharan [7].

We introduce new concepts called CTVC set and CTVC number of a graph. We denote it by $\alpha_\ast(G)$. We prove that $\beta_0(G) + \alpha_\ast(G) = n$ or $n + 1$. Where $n =$ number of vertices of $G$. We prove some theorems about removing a vertex from the graph.

We assume that our graphs are finite, simple and undirected. If $G$ is a graph then $V(G)$ will denote the vertex set of $G$ and $E(G)$ will denote the edge set of $G$.

2. **RESULTS AND DISCUSSION**

**Definition 2.1 (Colour Transversal Vertex Covering Set)**

Let $G$ be a graph. A subset $T$ of $V(G)$ is said to be a colour transversal vertex covering set of $G$ if

1. $T$ is a transversal of the colour classes of some chromatic colouring of $G$ and
2. $T$ is a vertex covering set of $G$

This set is also called CTVC set of $G$.

A CTVC set with minimum cardinality is called a minimum CTVC set or $\alpha_\ast$ set. The cardinality of an $\alpha_\ast$ set is called the colour transversal vertex covering number (or CTVC number) of the graph $G$ and it is denoted as $\alpha_\ast(G)$.

Note that for any graph $G$ and for any chromatic colouring of $G$, $V(G)$ is always a CTVC set. Thus a CTVC set always exists.

**Example 2.2**

Consider the cycle graph $C_5$ with vertices $v_1, v_2, v_3, v_4, v_5$
Consider the chromatic colouring which assigns colour - 1 to \( v_1 \& v_3 \), colour - 2 to \( v_2 \& v_5 \) and colour - 3 to \( v_4 \). Then the set \( S = \{ v_1, v_2, v_4 \} \) is a CTVC set of \( G \).

\[ \therefore \alpha_s (C_5) = 3 \]

Note that \( \alpha_0 (C_5) = 3 \)

**Remark 2.3**

We may note that for a given chromatic colouring of \( G \) there may not be a transversal corresponding to colour classes which is an independent set.

In fact it may happen that for any chromatic colouring of \( G \) such a set does not exists.

For example, consider the cycle graph \( C_5 \) again. In this graph it is impossible to have a set which is a transversal for some chromatic colouring and which is also an independent set. Because in this case a transversal must have atleast three vertices but the size of the maximum independent set of \( C_5 \) = 2

In general, If for any graph \( G \), \( \beta_0(G) < \chi(G) \) then there is no transversal which is an independent set. However it may happen that \( \chi(G) \leq \beta_0(G) \) but there is no transversal which is an independent set.

For example, consider the star graph with four vertices

\[ \chi(G) = 2, \quad \beta_0(G) = 3 \]

However there is no transversal which is an independent set.

**Example 2.4**

Consider the path graph with four vertices \( v_1, v_2, v_3, v_4 \)

\[ \chi(G) = 2, \quad \beta_0(G) = 3 \]

However there is no transversal which is an independent set.
Consider the chromatic colouring which assigns colour $-1$ to $v_1$& $v_3$ and colour $-2$ to $v_2$& $v_4$. Then obviously the set $\{v_1, v_4\}$ is a transversal which is also an independent set.

**Definition 2.5 (Dominator Colouring) [3]**

Let $G$ be a graph. A proper colouring $f$ of $G$ is said to be a dominator colouring if every colour class is a single vertex or it is completely dominated by some other colour class.

**Definition 2.6 (Colour Transversal) [7]**

Let $G$ be a graph and $C_1, C_2, \ldots, C_k$ be the colour classes of some proper colouring of $G$. A subset $T$ of $V(G)$ is said to be a colour transversal with respect to this colouring if $T \cap C_i \neq \emptyset$, $\forall i = 1, 2, \ldots, k$.

**Proposition 2.7**

Let $G$ be a graph. If a proper colouring of $G$ is a dominator colouring then there does not exist a colour transversal which is an independent set.

**Proof**

Let $G$ be a graph. Let $\{C_1, C_2, C_3, \ldots, C_k\}$ be the set of all colour classes corresponding to this proper colouring.

Suppose there is an independent set $S = \{v_1, v_2, \ldots, v_k\}$ which is a colour transversal of this colour classes. If $\forall i$, $\{v_i\}$ is a colour class then each $v_i$ is adjacent to each $v_j$ and therefore the subgraph induced by the vertices of $S$ is a complete subgraph.

This is a contradiction.

Therefore there is a colour class say $C_1$ which is not a singleton set. Let $v$ and $u$ be two distinct vertices of $C_1$. Then $u$ is completely dominated by some colour class say $C_j$. Therefore $u$ is adjacent to every vertex of $C_j$. Similarly for every other vertex of $C_1$ this happens.

$\therefore$ It is impossible to get a transversal which is an independent set.

**Proposition 2.8 [7]**

Let $G$ be a graph and $S$ be an independent subset of $G$ which is not a maximal independent subset of $G$. Then there is a chromatic colouring of $G$ in which $V(G) - S$ is a colour transversal for that colouring.

**Proof**

Let $f$ be any chromatic colouring of $G$. If $V(G) - S$ is a colour transversal for this colouring then the result is proved.

So, suppose $V(G) - S$ is not a colour transversal for this colouring. Then there is a colour class $C$ of this colouring such that $C \subseteq S$. Now, $S$ is not a maximal independent set. Therefore $\exists$ a vertex $z$ which is not in $S$ and it is not adjacent to any vertex of $S$. Let $C'$ be the colour class such that $z \in C'$. Suppose $C' = \{z\}$ then $z$ has neighbours in every other colour class. In particular, $z$ has neighbour in $C$. This implies that $z$ is adjacent to some vertex of $S$.

This is a contradiction.

$\therefore$ $C'$ contains at least two vertices one of which is $z$. Now define a new colouring $f'$ as follows.

$$f'(x) = f(x) \quad \text{if} \quad x \neq z \quad \& \quad f'(z) = f(t) \quad \text{where} \quad t \in C$$

Then $f'$ is a chromatic colouring of $f$ in which $V(G) - S$ is a colour transversal.

**Theorem 2.9**

Let $G$ be a graph with $n$ vertices. Then either $\beta_0(G) + \alpha_*(G) = n$ or $\beta_0(G) + \alpha_*(G) = n + 1$.

**Proof**

Suppose there is a maximum independent set $T$ such that $S = V(G) - T$ is a colour transversal for some chromatic colouring of $G$. Then $S$ is a colour transversal vertex covering set of $G$. 

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Claim
S is a minimum CTVC set of G

Proof of the Claim
Suppose S is not a minimum CTVC set of G. Let S₁ be a minimum CTVC set of G.

Then | S₁ | < | S |

Then | T | < | V(G) - S₁ | and V(G) - S₁ is an independent subset of G because S₁ is a vertex covering set of G.

This is a contradiction because T is a maximum independent subset of G. Thus, S must be a minimum CTVC set of G & therefore \( \alpha_s(G) = | S | \)

Obviously, \( \beta_0(G) + \alpha_s(G) = n \)

Suppose for any maximum independent set T, V(G) – T is not a colour transversal for any chromatic colouring of G.

Let T be any maximum independent subset of G. Let x ∈ T & consider the set T₁ = T – {x}. Then T₁ is an independent set which is not maximal.

By the above proposition, there is a chromatic colouring \( f \) of G such that S₁ = V(G) – T₁ is a colour transversal for this colouring. Since T₁ is an independent set, S is a vertex covering set. So, S is a CTVC set.

Claim
S is a minimum CTVC set

Proof of the Claim
Suppose S is not a minimum CTVC set. Let S₁ be a minimum CTVC set of G.

Then | S₁ | < | S |

Now, let T' = V(G) – S₁ Then T' is an independent set & | T' | > | T₁ |

Since, T' is an independent set | T' | = | T₁ | + 1

∴ T' is a maximum independent set such that S₁ = V(G) – T' is a colour transversal.

This is a contradiction.

Thus S = V(G) – T₁ is a minimum CTVC set of G.

i.e. \( \alpha_s(G) = | S | \)

Note that, | S | = n - \( \beta_0(G) \) + 1

Thus, \( \alpha_s(G) = n - \beta_0(G) + 1 \)

∴ \( \alpha_s(G) + \beta_0(G) = n + 1 \)

Corollary 2.10
Let G be a graph. Then, \( \alpha_0(G) = \alpha_s(G) \) or \( \alpha_0(G) = \alpha_s(G) - 1 \)

Proof
Suppose \( \alpha_s(G) + \beta_0(G) = n \)

Also, \( \alpha_0(G) + \beta_0(G) = n \)

∴ \( \alpha_0(G) = \alpha_s(G) \)

Suppose \( \alpha_s(G) + \beta_0(G) = n + 1 \)

∴ \( \alpha_s(G) - 1 + \beta_0(G) = n \)

Since, \( \alpha_0(G) + \beta_0(G) = n \)

\( \alpha_0(G) = \alpha_s(G) - 1 \)
Corollary 2.11

Let G be a graph. Then $\alpha_0(G) = \alpha_*(G)$ iff there is a maximum independent set $T$ of $G \ni V(G) - T$ is a colour transversal for some chromatic colouring of G.

Example 2.12

Consider the cycle graph $C_4$ then $\alpha_0(G) = 2$ & $\alpha_*(G) = 3$

For this graph, $\alpha_0(G) = \alpha_*(G) - 1$

Consider the cycle graph $C_5$ then $\alpha_0(G) = 3$ & $\alpha_*(G) = 3$

For this graph, $\alpha_0(G) = \alpha_*(G)$

Note 2.13 (Vertex Removal from a Graph)

Let G be a graph and $v \in V(G)$. Consider the subgraph $(G - v)$ also consider the numbers $\alpha_*(G)$ & $\alpha_*(G - v)$

We may ask the following question

What is the relation between $\alpha_*(G)$ & $\alpha_*(G - v)$?

We have the following proposition.

Theorem 2.14

Let G be a graph and $v \in V(G)$. Suppose $\chi(G - v) = \chi(G)$ then $\alpha_*(G - v) \leq \alpha_*(G)$

Proof

Let S be a minimum CTVC set of G with respect to some chromatic coloring f of G. Since $\chi(G - v) = \chi(G)$, $[v]$ is not a colour class for this chromatic colouring of G. Consider the function g which is restriction of f on G - v then g is a chromatic colouring of G - v because $\chi(G - v) = \chi(G)$.

Case 1: Suppose $v \notin S$

Then obviously S is a colour transversal for the chromatic colouring g of (G - v) because g uses the same colours as the f.

Also S is a vertex covering set of (G - v).

$\therefore$ S is a CTVC set of (G - v) (w.r.t. the chromatic colouring g)

$\therefore \alpha_*(G - v) \leq |S| = \alpha_*(G)$

Case 2: Suppose $v \in S$

Then $S - [v]$ is a vertex covering set of (G - v) but it need not be a colour transversal w.r.t. the colouring g. Let u be a vertex of G - v which has the same colour as v ([v] is not a colour class in f).

Let $S_1 = (S - [v]) \cup \{u\}$

Then $S_1$ is a CTVC set of (G - v).

$\therefore \alpha_*(G - v) \leq |S_1| = |S| = \alpha_*(G)$

Remark 2.15

It can be observed from example – 1 that $\alpha_*(C_5) = 3$ while $\alpha_*(C_5 - v_1) = 2$, $\alpha_*(C_5 - v_5) = 2$

Here, $\alpha_*(G - v) < \alpha_*(G)$

It can be observed from example – 2 that $\alpha_*(P_4) = 2$ while $\alpha_*(P_4 - v_1) = 2$, $\alpha_*(P_4 - v_2) = 2$

Here, $\alpha_*(G - v) = \alpha_*(G)$

Theorem 2.16

Let G be a graph and $v \in V(G)$. If $\alpha_*(G - v) < \alpha_*(G)$ then $\alpha_*(G - v) = \alpha_*(G) - 1$

Proof

Suppose that $\chi(G - v) < \chi(G)$ then $\chi(G - v) = \chi(G) - 1$
Let $S_1$ be minimum CTVC set of $(G - v)$ with respect to some chromatic colouring $f$ of $(G - v)$. Suppose this colouring has used colours $1, 2, \ldots, k - 1$.

If we assign any of this colour to $v$ then it will not be a proper colouring because $\chi(G - v) < \chi(G)$.

Therefore a new colour says $k$ must be assigned to vertex $v$ to get a new chromatic colouring $f'$ of $G$ as follows

$$f'(v) = k$$ and

$$f'(w) = f(w) \text{ if } w \neq v$$

The set $S_1$ may or may not be a vertex covering set of $G$ but it is certainly not a colour transversal for this colouring $f'$ of $G$. Also it can not be a colour transversal for any chromatic colouring of $G$ because it will imply that the chromatic number of $G = k - 1$.

If $S = S_1 \cup \{v\}$ then $S$ is both a colour transversal & a vertex covering set of $G$.

Since $\alpha_0(G - v) < \alpha_0(G)$, $S$ must be a minimum CTVC set of $G$.

$$\therefore \alpha_0(G) = |S| = |S_1| + 1 = \alpha_0(G - v) + 1$$

Now suppose $\chi(G - v) = \chi(G)$

Let $S_1$ be minimum CTVC set of $(G - v)$ with respect to some chromatic colouring $f$ of $(G - v)$.

Since $\chi(G - v) = \chi(G)$, $\{v\}$ is not a colour class in any chromatic colouring of $G$.

Let $g$ be a chromatic colouring of $G$ then the restriction of $G$ on $(G - v)$ is the chromatic colouring $f$. In this colouring the colour of $v$ will also appear as colour of some other vertex of $G$.

$$\therefore S_1$$ is a colour transversal for this colouring $g$.

Since $\alpha_0(G - v) < \alpha_0(G)$, $S_1$ can not be a vertex covering set of $G$. Let $S = S_1 \cup \{v\}$ then obviously $S$ is a vertex covering set of $G$ and it is also a colour transversal with respect to chromatic colouring of $G$. Since $\alpha_0(G - v) < \alpha_0(G)$, the set $S$ must be minimum.

Thus, $\alpha_0(G) = |S| = |S_1| + 1 = \alpha_0(G - v) + 1$

**Proposition 2.17**

Let $G$ be a graph and $v \in V(G)$. If $\alpha_0(G - v) < \alpha_0(G)$ and $\alpha_0(G) = \alpha_0(G)$ then $\alpha_0(G - v) < \alpha_0(G)$

**Proof**

Suppose $\alpha_0(G - v) = \alpha_0(G)$

Now, $\alpha_*(G - v) = \alpha_*(G) - 1$

$$= \alpha_0(G) - 1$$

$$< \alpha_0(G) = \alpha_0(G - v)$$

$$\therefore \alpha_0(G - v) < \alpha_0(G)$$

This is a contradiction

$$\therefore \alpha_0(G - v) < \alpha_0(G)$$

**Corollary 2.18**

Let $G$ be a graph and $v \in V(G)$. If $\alpha_0(G - v) < \alpha_0(G)$ & $\alpha_0(G) < \alpha_0(G)$ then $\alpha_*(G - v) < \alpha_*(G)$

**Proof**

Suppose $\alpha_*(G - v) = \alpha_*(G)$

Then, $\alpha_*(G - v) = \alpha_*(G) > \alpha_0(G - v)$

Now, $\alpha_0(G) = \alpha_*(G) - 1$ and $\alpha_0(G - v) = \alpha_0(G) - 1$

$$\therefore \alpha_0(G - v) = \alpha_*(G - v) - 2$$
Which is not possible
\[ \therefore \alpha_0(G - v) < \alpha_0(G) \]

**Proposition 2.19**

If \( \alpha_0(G - v) > \alpha_0(G) \) then \( \alpha_0(G - v) = \alpha_0(G) = \alpha_0(G) \)

**Proof**

First we prove that \( \alpha_0(G) = \alpha_0(G) \)

Suppose \( \alpha_0(G) < \alpha_0(G) \)

Then \( \alpha_0(G - v) - \alpha_0(G - v) = \alpha_0(G - v) - \alpha_0(G) + \alpha_0(G) - \alpha_0(G - v) \)

\[ \geq 1 + 1 + 0 = 2 \]

\[ \therefore \alpha_0(G - v) - \alpha_0(G - v) \geq 2 \]

Which is not possible. Thus, \( \alpha_0(G) = \alpha_0(G) \)

Suppose \( \alpha_0(G - v) < \alpha_0(G) \)

Then \( \alpha_0(G - v) - \alpha_0(G - v) = \alpha_0(G - v) - \alpha_0(G) + \alpha_0(G) - \alpha_0(G - v) ( \therefore \alpha_0(G) = \alpha_0(G) ) \)

\[ \geq 1 + 1 = 2 \]

Again this is a contradiction.

\[ \therefore \alpha_0(G - v) = \alpha_0(G) \]

**Remark 2.20**

From the above proposition it follows that if \( \alpha_0(G - v) > \alpha_0(G) \) then every minimum CTVC set of \( G \) does not contain \( v \) because such a set is always a minimum vertex covering set of \( G \) (\( \therefore \alpha_0(G) = \alpha_0(G) \) is a minimum vertex covering set of \( G \) and since \( \alpha_0(G - v) = \alpha_0(G) \) no minimum vertex covering set can contain vertex \( v \).)

3. **CONCLUDING REMARK**

We have proved in theorem 2 that if \( \chi(G - v) = \chi(G) \) then \( \alpha_0(G - v) \leq \alpha_0(G) \) however we do not know if \( \chi(G - v) < \chi(G) \) then \( \alpha_0(G - v) \leq \alpha_0(G). \)

We Present the following conjecture.

3.1 **Conjecture**

If \( \chi(G - v) < \chi(G) \) then \( \alpha_0(G - v) \leq \alpha_0(G). \)

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AUTHORS’ BIOGRAPHY

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