Schur Convexity Condition for Novel Ratio of Difference of Means

Sreenivasa Reddy Perla
The Oxford College of Engg
Dept of Mathematics, VTU
Bangalore, Karnataka. India.
srreddy_sri@yahoo.co.in

K M Nagaraja
JSSATE
Dept of Mathematics, VTU
Bangalore, Karnataka. India.
nagkmn@gmail.com

Padmanabhan S
RNSIT
Dept of Mathematics, VTU
Bangalore, Karnataka. India.
padanabhanrnsit@gmail.com

K M Nagaraja
JSSATE
Dept of Mathematics, VTU
Bangalore, Karnataka. India.
nagkmn@gmail.com

Padmanava Samanta
Dept of Mathematics
Berhampur University,
Odissa. India.

Abstract: In this paper, we study the different kind of convexities like Schur, Schur Geometric and Schur Harmonic convexities for novel ratio of difference of means.

Keywords: Schur convexity, Schur harmonic convexity, Schur Geometric convexity, ratio of difference of means and inequality.

1. INTRODUCTION

The well-known means in literature such as arithmetic mean, geometric mean harmonic mean and contra harmonic mean are presented by pappus of Alexandria. In Pythagorean School on the basis of proportion and also some of the other means like Heron mean and Centriodal Mean are defined as follows [1,2] and some interesting results on above said means are discussed in [3-8].

For two positive real numbers a & b :

Arithmetic Mean = \( A(a,b) = \frac{a + b}{2} \)  
Geometric Mean = \( G(a,b) = \sqrt{ab} \)

Harmonic Mean = \( H(a,b) = \frac{2ab}{a + b} \)  
Contra Harmonic Mean = \( C(a,b) = \frac{a^2 + b^2}{a + b} \)

Heron mean = \( H_e(a,b) = \frac{a + \sqrt{ab} + b}{3} \)  
and Centriodal Mean \( C_d(a,b) = \frac{2(a^2 + ab + b^2)}{3(a + b)} \)

Jamal Rooin and Mehdi Hassni [9], introduced the homogeneous functions \( f(x) \) and \( g(x) \), also established some convexity results and refinements to Ky-Fan-type inequalities.where

\[ f(x) = \frac{a^x - b^x}{c^x - d^x} \quad \text{and} \quad g(x) = \ln \frac{a^x - b^x}{c^x - d^x} \quad \text{for} \quad x \in (-\infty, \infty) \quad \text{and} \quad a > b \geq c > d. \]

In [10], authors studied the convexity(concavity) of the following ratio of difference of means.

\[ M_{CAGH}(a,b) = \frac{C(a,b) - A(a,b)}{G(a,b) - H(a,b)} \quad \quad M_{AH,GH}(a,b) = \frac{A(a,b) - H_e(a,b)}{G(a,b) - H(a,b)} \]
\[
M_{CH,GH}(a,b) = \frac{C(a,b) - H_e(a,b)}{G(a,b) - H(a,b)}
\]
\[
M_{C,AGH}(a,b) = \frac{C_d(a,b) - A(a,b)}{G(a,b) - H(a,b)}
\]
\[
M_{CAH,G}(a,b) = \frac{C(a,b) - A(a,b)}{H_e(a,b) - G(a,b)}
\] and
\[
M_{CC,GH}(a,b) = \frac{C(a,b) - C_d(a,b)}{G(a,b) - H(a,b)}
\]

Some fruitful results related to Schur convexities were also found in [11-22].

In this paper, we study the Schur, Schur harmonic and Schur geometric convexity of ratio of means \( M_{CAGH}(a,b), M_{AH,GH}(a,b), M_{CH,GH}(a,b) \) & \( M_{CAH,G}(a,b) \) and some applications of these ratio of difference of means.

2. PRELIMINARY RESULTS AND DEFINITIONS

In 1923, the Schur Convex function was introduced by I Schur, and proved many important applications to analytic inequalities. In 2003, X. M. Zhang propose the concept of Schur-geometrically convex set which is an extension of Schur-convexity function. In recent years, the Schur convexity, Schur geometrically convexity and Schur harmonic convexity have attracted the attention of a considerable number of mathematicians ([11]-[21]). For convenience of readers, we recall some definitions as follows:

**Definition 2.1** [3,7] Let \( x = (x_1, x_2, x_3, \ldots, x_n) \) and \( y = (y_1, y_2, y_3, \ldots, y_n) \in R^n \)

1. Let \( x \) is said to be majorized by \( y \) (in symbol \( x < y \)) if \( \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \) for \( k = 1, 2, 3, \ldots n \) and \( \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i \) where \( x_i \geq \ldots \geq x_n \) and \( y_i \geq \ldots \geq y_n \) are rearrangement of \( x \) and \( y \) in descending order.

2. \( \Omega \subseteq R^n \) The function \( \varphi : \Omega \rightarrow R^n \) is said to be schur convex function on \( \Omega \) if \( x < y \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). \( \varphi \) is said to be a Schur concave function on \( \Omega \) if and only if

\( -\varphi \) is Schur convex.

**Definition 2.2** [22] Let \( x = (x_1, x_2, x_3, \ldots, x_n) \) and \( y = (y_1, y_2, y_3, \ldots, y_n) \in R^n_+ \), \( \Omega \subseteq R^n_+ \) is called geometrically convex set if \( x^\alpha y^\beta \in R^n \) for all \( x \) and \( y \) where \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta = 1 \). Let \( \Omega \subseteq R^n_+ \). The function \( \varphi : \Omega \rightarrow R^n_+ \) is said to be schur geometrically convex function on \( \Omega \) if \( \ln x \leq \ln y \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). Let \( \varphi \) is said to be a Schur geometrically concave function on \( \varphi \) if and only if \( -\varphi \) is Schur geometrically convex.

**Definition 2.3** [3,7] The set \( \Omega \subseteq R^n \) is called symmetric set if \( x \in \Omega \) implies \( px \in \Omega \) for every \( n \times n \) permutation matrix \( p \). The function \( \varphi : \Omega \rightarrow R^n \) is said to be symmetric if every permutation matrix \( p \); \( \varphi(px) = \varphi(x) \) for all \( x \in \Omega \).

we introduce the ratio of difference of means as follows:

\[
\frac{M_{CAGH}}{M_{AH,GH}}(a,b) = \frac{C(a,b) - A(a,b)}{A(a,b) - H_e(a,b)}
\]
\[
\frac{M_{CC,GH}}{M_{C,AGH}}(a,b) = \frac{C(a,b) - C_d(a,b)}{C_d(a,b) - A(a,b)}
\]
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\[
\begin{align*}
\frac{M_{C_{AGH}}(a,b)}{M_{A_{GH}}(a,b)} &= \frac{C_d(a,b)-A(a,b)}{A(a,b)-H_x(a,b)} \\
\frac{M_{C_{AGH}}(a,b)}{M_{A_{GH}}(a,b)} &= \frac{C(a,b)-C_d(a,b)}{A(a,b)-H_x(a,b)} \\
\frac{M_{C_{AGH}}(a,b)}{M_{A_{GH}}(a,b)} &= \frac{C(a,b)-C_d(a,b)}{A(a,b)-H_x(a,b)}
\end{align*}
\]

**Lemma 2.1.** Let \( \Omega \subseteq \mathbb{R}^n \) \( \varphi : \Omega \rightarrow R \) is symmetric and convex function. Then \( \varphi \) is Schur convex on \( \Omega \).

**Lemma 2.2.** For \( a > b \geq c > d \), the function \( f(x) = \frac{a^x - b^x}{c^x - d^x} \) where \( x \in (-\infty, \infty) \) is

i) Convex, if \( ad - bc > 0 \)

ii) Concave if \( ad - bc < 0 \)

iii) Equality holds if \( ad - bc = 0 \).

**Lemma 2.3.** Let \( \Omega \subseteq \mathbb{R}^n \) be symmetric with non empty interior convex set and let \( \varphi : \Omega \rightarrow R \) be continuous on \( \Omega \) and differentiable on \( \Omega^0 \) if \( \varphi \) is symmetric on \( \Omega \) and for any \( x = (x_1, x_2, x_3, \ldots, x_n) \in \Omega^0 \). Then \( \varphi \) is

(i) Schur convex (concave) if \( s = (x_1 - x_2) \left( \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \geq 0(\leq 0) \).

(ii) Schur geometrically convex (concave) if \( s = (\ln x_1 - \ln x_2) \left( x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2} \right) \geq 0(\leq 0) \).

(iii) Schur harmonically convex (concave) if \( s = (a - b) \left( x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2} \right) \geq 0(\leq 0) \).

### 3. Schur Properties on Ratio of Difference of Mean

In this section, the Schur, Schur geometric and Schur harmonically convexities on ratio of difference of mean are established by finding the partial derivatives.

**Theorem 3.1.** The ratio of difference of means \( \frac{M_{CC_{GH}}}{M_{C_{AGH}}} \) is

(i) Schur convex.

(ii) Schur geometrically convex.

(iii) Schur harmonically convex is for all \( a \geq b \).

**Proof.** Let \( \frac{M_{CC_{GH}}(a,b)}{M_{C_{AGH}}(a,b)} = \frac{C(a,b) - C_d(a,b)}{C_d(a,b) - A(a,b)} \)

From lemma 2, Consider, \( f(a,b) = CA - C_d^2 = \left( \frac{a^2 + b^2}{a + b} \right) \left( \frac{a + b}{2} \right) - \frac{4}{9} \left( \frac{a^2 + ab + b^2}{a + b} \right)^2 \)
By finding the partial derivatives of \( f(a,b) \) and with simple manipulation gives

\[
\frac{\partial f}{\partial a} = a - \frac{8}{9} \left( \frac{a^2 + ab + b^2}{(a+b)^3} \right) a(a+2b) \quad \text{and} \quad \frac{\partial f}{\partial b} = b - \frac{8}{9} \left( \frac{a^2 + ab + b^2}{(a+b)^3} \right) b(2a+b) \quad \text{(1)}
\]

Proof of (i), from eqn (1), we have

\[
\frac{\partial f}{\partial a} - \frac{\partial f}{\partial b} = (a-b) \left( 1 - \frac{8(a^2 + ab + b^2)}{9(a+b)^2} \right) = (a-b) \left( \frac{a^2 + 10ab + b^2}{9(a+b)^2} \right)
\]

Then \( s = (a-b) \left( \frac{\partial f}{\partial a} - \frac{\partial f}{\partial b} \right) = (a-b)^2 \left( \frac{a^2 + 10ab + b^2}{9(a+b)^2} \right) \geq 0 \) for all \( a \) & \( b \)

This verifies the condition for Schur convexity.

Proof of (ii), from eqn (1), we have

\[
a \frac{\partial f}{\partial a} - b \frac{\partial f}{\partial b} = (a^2 - b^2) - \frac{8}{9} \left( \frac{a^2 + ab + b^2}{(a+b)^3} \right) \left( a(b^2 - 1) + 2ab(a-b) \right) \geq 0 \text{ for all } a \text{ & } b
\]

Then \( s = \ln a - \ln b \left( a \frac{\partial f}{\partial a} - b \frac{\partial f}{\partial b} \right) = (\ln a - \ln b) \left( \frac{a-b}{9(a+b)^2} \right) \left( a^4 + b^4 + 14a^2b^2 + 2ab^3 \right) \geq 0 \)

This verifies the condition for Schur geometrically convexity.

Proof of (iii), As above we have

Then \( s = (a-b) \left( a^2 \frac{\partial f}{\partial a} - b^2 \frac{\partial f}{\partial b} \right) = (a-b) \left( \frac{a^2 + 10ab + b^2}{9(a+b)^2} \right) \geq 0 \) for \( a \) & \( b \)

This verifies the condition for Schur Harmonically convex.

Thus the proof of theorem 3.1 is completed.

**Theorem 3.2.** The ratio of difference of means \( \frac{M_{CC, GH}}{M_{AH, GH}} \) is

(i) Schur convex.

(ii) Schur geometrically convex.

(iii) Schur harmonically convex is for all \( a \geq b \).

Proof: Let \( M_{CC, GH} \) = \( C(a,b) - C_d(a,b) \)

\[
\frac{\partial}{\partial a} \left( \frac{a+b+\sqrt{ab}}{3} \right) - \left( \frac{a+b}{2} \right)^2 \left( \frac{a^2 + ab + b^2}{a+b} \right) - \frac{\sqrt{ab}}{3} \left( \frac{a+b - 2ab}{a+b - \sqrt{ab}} \right)
\]

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By finding the partial derivatives of \( f(a,b) \) and with simple manipulation gives

\[
\frac{\partial f}{\partial a} = \frac{\sqrt{ab}}{3} \left(1 - \frac{2b^2}{(a+b)^2} - \frac{\sqrt{b}}{2\sqrt{a}}\right) + \left(a + b - \frac{2ab}{a+b} - \sqrt{ab}\right) \frac{\sqrt{b}}{6\sqrt{a}}
\]

\[
a \frac{\partial f}{\partial a} = \frac{\sqrt{ab}}{6} \left(2a - \frac{4ab^2}{(a+b)^2} - \sqrt{ab}\right) + \left(a + b - \frac{2ab}{a+b} - \sqrt{ab}\right)
\]

similarly \( b \frac{\partial f}{\partial b} = \frac{\sqrt{ab}}{6} \left(2b - \frac{4a^2b}{(a+b)^2} - \sqrt{ab}\right) + \left(a + b - \frac{2ab}{a+b} - \sqrt{ab}\right) \) \hspace{1cm} (2)

Proof of (i), from eqs (2) & (3) we have

\[
Then \ s = (a-b)\left(\frac{\partial f}{\partial a} - \frac{\partial f}{\partial b}\right) = \frac{(a-b)^2}{6(a+b)(\sqrt{ab})} \left(4ab + 2\sqrt{ab}(a+b) - a^2 - b^2\right) \geq 0 \text{ for } a \neq b.
\]

This verifies the condition for Schur convex.

Proof (ii), from eqs (2) & (3) we have

\[
a \frac{\partial f}{\partial a} - b \frac{\partial f}{\partial b} = \frac{\sqrt{ab}}{6} \left(2a - 2b - \frac{4ab(b-a)}{(a+b)^2}\right)
\]

Consider, \( s = (\ln a - \ln b) \left(a \frac{\partial f}{\partial a} - b \frac{\partial f}{\partial b}\right) = (\ln a - \ln b) \frac{\sqrt{ab}(a-b)}{3} \left(1 + \frac{2ab}{(a+b)^2}\right) \geq 0 \text{ for } a \neq b
\]

This verifies the condition for Schur geometrically convex.

Proof of (iii), from eqs (2) & (3) we have

\[
a^2 \frac{\partial f}{\partial a} - b^2 \frac{\partial f}{\partial b} = \frac{\sqrt{ab}}{6} (a-b) \left(3a + 3b - 2\sqrt{ab} - \frac{2ab}{(a+b)}\right)
\]

Then \( s = (a-b) \left(a^2 \frac{\partial f}{\partial a} - b^2 \frac{\partial f}{\partial b}\right) = \frac{\sqrt{ab}}{6} \frac{(a-b)}{(a+b)} \left(3(a^2 + b^2) + 4ab - 2\sqrt{ab}(a+b)\right) \geq 0 \text{ for } a \neq b
\]

This verifies the condition for Schur harmonically convex.

Thus the proof of theorem 3.2 is completed.

**Theorem 3.3.** The ratio of difference of means \( \frac{M_{CAGH}}{M_{AH,GH}} \) is

(i) Schur convex.

(ii) Schur geometrically convex.

(iii) Schur harmonically convex \( \text{ for all } a \geq b \).

Proof: Similar argument as discussed in theorem 3.1 and 3.2 gives the proof of theorem 3.3.
REFERENCES