Expansion Formula for the Multivariable $A$-Function Involving Generalized Legendre’s Associated Function

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Abstract: The authors have established a new expansion formula for multivariable $A$-function due to Gautam et. al. [3] in terms of products of the multivariable $A$-function and the generalized Legendre’s associated function due to Meulenbeld [4]. Some special cases are given in the last.

Keywords: Multivariable $A$-function, Generalized Legendre’s associated function, Multivariable $H$-function.

(2000 Mathematics subject classification: 33C99)

1. INTRODUCTION

Gautam and Goyal [3] defined and represented the multivariable $A$-function as follows:

$$A[z_1, \ldots, z_r] = A_{m,n,m',n';\ldots;m',n'}^{p,q;p',q';\ldots;p',q'}$$

$$= \frac{1}{(2\pi i)} \int_{L_i} \cdots \int_{L_i} \theta_i(s_i) \cdots \theta_r(s_r) \Phi(s_1, \ldots, s_r) z_1^{s_1} \cdots z_r^{s_r} ds_1 \cdots ds_r \quad (1.1)$$

Where $\omega = \sqrt{-1}$;

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m} \Gamma(d_j^{(i)} - D_j^{(i)} s_i) \prod_{j=1}^{n} \Gamma(1 - c_j^{(i)} + C_j^{(i)} s_i)}{\prod_{j=m+1}^{q} \Gamma(1 - d_j^{(i)} + D_j^{(i)} s_i) \prod_{j=n+1}^{p} \Gamma(c_j^{(i)} - C_j^{(i)} s_i)}$$

$$\forall i \in \{1, \ldots, r\} \quad (1.2)$$

$$\Phi(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} A_j^{(i)} s_i) \prod_{j=1}^{m} \Gamma(b_j - \sum_{i=1}^{r} B_j^{(i)} s_i)}{\prod_{j=m+1}^{q} \Gamma(a_j - \sum_{i=1}^{r} A_j^{(i)} s_i) \prod_{j=n+1}^{p} \Gamma(1 - b_j + \sum_{i=1}^{r} B_j^{(i)} s_i)}$$

Here $m, n, p, q, m', n', p', q'$ are non-negative integers and all $a_j, b_j, c_j^{(i)}, d_j^{(i)} s, A_j^{(i)} s, B_j^{(i)} s$ and $c_j^{(i)}, d_j^{(i)}$ are complex numbers.

The multiple integral defining the $A$-function of $r$-variables converges absolutely if
\begin{equation}
\left| \arg(\Omega_i) \right| \zeta_k^* < \frac{\pi}{2} \eta, \zeta_k^* = 0, \eta > 0
\end{equation}

$$\Omega_i = \prod_{j=1}^{p} \left\{ A_j^{(i)} \right\}^{\frac{\alpha_j}{\beta_j}} \prod_{j=1}^{q} \left\{ B_j^{(i)} \right\}^{\frac{\beta_j}{\alpha_j}} \prod_{j=1}^{n} \left\{ C_j^{(i)} \right\}^{-\gamma_j^{(i)}}, \forall i \in \{1, \ldots, r\}$$

$$\zeta_i^* = \int_m \left[ \sum_{j=1}^{p} A_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)} + \sum_{j=1}^{n} D_j^{(i)} - \sum_{j=1}^{n} C_j^{(i)} \right], \forall i \in \{1, \ldots, r\}$$

$$\eta_i = \operatorname{Re} \left[ \sum_{j=1}^{n} A_j^{(i)} - \sum_{j=1}^{m} A_j^{(i)} + \sum_{j=1}^{n} B_j^{(i)} + \sum_{j=1}^{m} B_j^{(i)} + \sum_{j=1}^{n} D_j^{(i)} + \sum_{j=1}^{m} D_j^{(i)} + \sum_{j=1}^{n} C_j^{(i)} - \sum_{j=1}^{m} C_j^{(i)} \right], \forall i \in \{1, \ldots, r\}$$

If we take $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}$ and $D_j^{(i)}$ as real and positive and $m = 0$, the $A$-function reduces to multivariable $H$-function of Srivastava and Panda [7]

In this paper we will evaluate an integral involving generalized associated Legendre’s function and the multivariable $A$-function due to Gautam [3] and apply it in deriving an expansion for the multivariable $A$-function in series of products of associated Legendre’s function and the multivariable $A$-function.

2. THE INTEGRAL

The integral to be evaluated is:

$$\int_{1}^{1-x} \frac{(1-x)^{\alpha_i} (1+x)^{\beta_i}}{\zeta_k^{(i)}} dx$$

$$\times A \left[ (1-x)^{\alpha_i} (1+x)^{\beta_i}, \ldots, (1-x)^{\alpha_i} (1+x)^{\beta_i} \right] dx$$

$$= 2^{\alpha_i + \beta_i + 1} \sum_{r=0}^{\alpha_i + \beta_i + 1} \frac{(-1)^r (v-u+k+1)}{(1-u+k+1)} A_{m,n+2(m',n',\ldots; m',n')}^{(\alpha_i, \beta_i)}$$

$$\left[ \begin{array}{c}
\frac{\hat{\alpha}_i (r, \ldots, \hat{\alpha}_i)}{\hat{\beta}_i (r, \ldots, \hat{\beta}_i)} \\
\frac{\hat{\alpha}_i (r, \ldots, \hat{\alpha}_i)}{\hat{\beta}_i (r, \ldots, \hat{\beta}_i)}
\end{array} \right]_{m,n}$$

$$\left( \begin{array}{c}
\frac{(-1)^{r+v} \beta_i}{\alpha_i} \\
\frac{(-1)^{r+v} \beta_i}{\alpha_i}
\end{array} \right)_{m,n}$$

$$\left( \begin{array}{c}
\frac{(-1)^{r+v} \beta_i}{\alpha_i} \\
\frac{(-1)^{r+v} \beta_i}{\alpha_i}
\end{array} \right)_{m,n}$$

(2.1)

The integral (2.1) is valid under the following set of conditions:

(i) $\alpha_i, \beta_i > 0; \forall i \in 1, 2, \ldots, r; k = \frac{u-v}{2}$ is a positive integer, $k$ is an integer $\geq 0$.

(ii) $\operatorname{Re} \left( \rho - u + \sum_{i=1}^{r} \alpha_i \frac{b_{i}^{(i)}}{\beta_{i}^{(i)}} \right) > -1; \operatorname{Re} \left( \sigma + v + \sum_{i=1}^{r} \beta_i \frac{b_{i}^{(i)}}{\beta_{i}^{(i)}} \right) > -1; (j = 1, 2, \ldots, m_i; i = 1, 2, \ldots, r)$

And the conditions given in (1.4) to (1.7) are also satisfied.

**Proof:** On expressing the multivariable $A$-function in the integrand as a multiple Mellin-Barnes type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral
**Expansion Formula for the Multivariable $A$-Function Involving Generalized Legendre’s Associated Function**

\[
= (2\pi w)^r \prod_{i=1}^{r} s_i(s_i) \sum_{i=1}^{r} \phi(s_i) z_i^{\tilde{s}_i}
\]

\[
\times \left\{ (1-x)^{\rho} (1+x)^{\sigma} \sum_{i=1}^{r} \sigma_{i}^{\frac{m-n}{2}} \sum_{i=1}^{r} \beta_{i}^{\frac{m-n}{2}} \right\}
\]

\[
P_{m,n}^{\mu,\nu}(x) dx \begin{bmatrix} d\xi_1 \ldots d\xi_r \end{bmatrix}
\]

On evaluating the $x$-integral with the help of the integral ([5], p. 343, eq. (38)):

\[
\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_{m,n}^{\mu,\nu}(x) dx
\]

\[
= 2^{\rho+\sigma-\frac{m-n}{2}} \Gamma \left( \rho-\frac{m}{2}+1 \right) \Gamma \left( \sigma+\frac{n}{2}+1 \right)
\]

\[
\times \Gamma \left( 1-m \right) \Gamma \left( \rho+\sigma-\frac{m-n}{2}+2 \right)
\]

\[
\times \beta_{\mu,\nu} \left( -k, n-m+k+1, \rho-\frac{m}{2}+1; 1-m, \rho-\sigma-\frac{m-n}{2}+2; 1 \right)
\]

Provided that $\text{Re} \left( \rho-\frac{m}{2} \right) > -1; \text{Re} \left( \sigma+\frac{n}{2} \right) > -1$ and interpreting the result with the help of (1.1), the integral (2.1) is established.

3. **Expansion Theorem**

Let the following conditions be established:

(i) $\beta_1, \ldots, \beta_r > 0; \alpha_1, \ldots, \alpha_r \geq 0 (or \beta_1, \ldots, \beta_r \geq 0; \alpha_1, \ldots, \alpha_r > 0)$;

(ii) $m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i = 1, \ldots, r)$ are non-negative integers where

\[
0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q^k \geq 0, 0 \leq n \leq p \text{ and the conditions given by (1.4) to (1.7) are also satisfied.}
\]

(iii) $\text{Re}(u) > -1, \text{Re}(v) > -1, \text{Re} \left( \rho-u + \sum_{i=1}^{r} \alpha_{i} \frac{b^{(i)}_{j}}{\beta_{j}^{(i)}} \right) > -1$;

\[
\text{Re} \left( \sigma+v + \sum_{i=1}^{r} \beta_{i} \frac{b^{(i)}_{j}}{\beta_{j}^{(i)}} \right) > -1; (j = 1, 2, \ldots, m_i; i = 1, 2, \ldots, r).
\]

Then the following expansion formula holds:

\[
(1-x)^{\rho} (1+x)^{\sigma} A \left[ (1-x)^{a_1} (1+x)^{b_1} z_1, \ldots, (1-x)^{a_r} (1+x)^{b_r} z_r \right]
\]

\[
= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_{N}}{N!\mu!\Gamma(1+v+N)\Gamma(1-u+\mu)}
\]
(3.1) Proof: Let

\[ f(x) = (1-x)^{m+\frac{n}{2}} (1+x)^{n+\frac{m}{2}} A \left[ (1-x)^{\alpha_1} (1+x)^{\beta_1}, \ldots, (1-x)^{\alpha_r} (1+x)^{\beta_r} \right] \]

\[ = \sum_{N=0}^{\infty} C_N^\nu P_{N-u-v}^{\mu,v} (x) \]  

Equation (3.2) is valid since \( f(x) \) is continuous and of bounded variation in the interval \((-1,1)\).

Now, multiplying both the sides of (3.2) by \( P_{N-u-v}^{\mu,v} (x) \) and integrating with respect to \( x \) from \(-1\) to \(1\); evaluating the L.H.S. with the help of (2.1) and on the R.H.S. interchanging the order of summation, using ([2], p. 176, eq. (75)) and then applying orthogonality property of the generalized Legendre’s associated functions ([5], p. 340, eq. (27)):

\[ \int_{-1}^{1} P_{k}^{\mu,v} (x) P_{N-u-v}^{\mu,v} (x) dx \]

Provided that \( \text{Re}(\mu), 1, \text{Re}(\nu) < 1 \); we obtain

\[ \mathcal{C}_k = \frac{2^{(\nu+\mu)} (2k-u+v+1) \Gamma(k-u+1) \mu! \Gamma(k-u+\mu)}{k! \Gamma(k+v+1)} \sum_{\mu=0}^{\infty} \left[ (-k)_\mu \Gamma(k-u+v+\mu+1) \right. \]

\[ = \frac{2^{(\nu+\mu)} (2k-u+v+1) \Gamma(k-u+1) \mu! \Gamma(k-u+\mu)}{k! \Gamma(k+v+1)} \sum_{\mu=0}^{\infty} \left[ (-k)_\mu \Gamma(k-u+v+\mu+1) \right. \]

Now on substituting the values of \( \mathcal{C}_k \) in (3.2), the result follows.

4. SPECIAL CASES

If in (2.1), we put \( m = 0 \), the multivariable \( A \)-function occurring in the left-hand side of these formulae would reduce immediately to multivariable \( H \)-function due to Srivastava et. al. [7] and we get result given by Saxena and Ramawat [6]

\[ (1-x)^{m+\frac{n}{2}} (1+x)^{n+\frac{m}{2}} H \left[ (1-x)^{\alpha_1} (1+x)^{\beta_1}, \ldots, (1-x)^{\alpha_r} (1+x)^{\beta_r} \right] \]

\[ = 2^{(\nu+\mu)} \sum_{N=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(2N-u+v+1) \Gamma(N-u+1) \Gamma(1+v-u+N+\mu)(-N)_\mu}{N! \mu! \Gamma(1+v+N) \Gamma(1-u+\mu)} \]

\[ = 2^{(\nu+\mu)} \sum_{N=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(2N-u+v+1) \Gamma(N-u+1) \Gamma(1+v-u+N+\mu)(-N)_\mu}{N! \mu! \Gamma(1+v+N) \Gamma(1-u+\mu)} \]

\[ P_{N-u-v}^{\mu,v} (x) H \]
Expansion Formula for the Multivariable $A$-Function Involving Generalized Legendre’s Associated Function

\[
\begin{bmatrix}
\begin{array}{c}
2^{\alpha_{1}}+r, 
\vdots 
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\beta_{1}, \ldots, \beta_{r} \end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r}, \gamma, \lambda, \mu),
(\alpha_{1}, \alpha_{2}, \ldots, \lambda, \mu),
(\alpha_{1}, \lambda, \mu),
(\gamma, \lambda, \mu),
(\lambda, \mu),
(\mu),
\end{array}
\end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{array}{c}
(\lambda, \mu),
(\gamma, \lambda, \mu),
(\lambda, \mu),
(\mu),
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r}, \gamma, \lambda),
(\alpha_{1}, \alpha_{2}, \ldots, \lambda, \mu),
(\alpha_{1}, \lambda, \mu),
(\gamma, \lambda, \mu),
(\lambda, \mu),
(\mu),
\end{array}
\end{bmatrix}
\end{bmatrix}
\]

Provided all the conditions given with (3.1) and the conditions ([7], p.252-253, eq. (c.4), (c.5) and (c.6)) are satisfied.

For $n = 0 = p, q = 0$, the multivariable $H$-function breaks up into a product of $r$ $H$-function and consequently, (4.1) reduces to

\[
(1-x)^{\mu/2} (1+x)^{\nu/2} \prod_{i=1}^{r} \left\{ H_{\mu,\nu}^{\mu,\nu} \right\} = 2^{\rho+\sigma} \sum_{N=0}^{\infty} \frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_{\mu}}{N!\Gamma(1+v+N)\Gamma(1-u+\mu)}
\]

\[
P_{x}^{u,v} \left( \frac{x}{N} \right) H_{\mu,\nu}^{\mu,\nu} \left[ \begin{array}{c}
\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r},
\beta_{1}, \beta_{2}, \ldots, \beta_{r},
\end{array} \right]
\]

For $r = 1$, (4.2) gives rise to the result due to Anandani [1].

REFERENCES


