(i, j)-Almost Continuity and (i, j)-Weakly Continuity in Fuzzy Bitopological Spaces

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Abstract: Fuzzy regular open (closed) sets, fuzzy almost continuous and fuzzy weakly continuous maps on fuzzy topological spaces have been studied in [1]. In the present paper we introduce the concepts of fuzzy (i, j)-regular open (closed) sets, fuzzy (i, j)-almost continuous and fuzzy (i, j)-weakly continuous maps on fuzzy bitopological spaces. Fuzzy (i, j)-semi regular and fuzzy (i, j)-regular spaces have also been studied.

Keywords: Fuzzy bitopological spaces, fuzzy regular open sets, fuzzy continuous mappings, fuzzy almost continuous mappings, fuzzy weakly continuous mappings.

1. INTRODUCTION


Fuzzy bitopological spaces have been introduced by Kandil [4] in 1989. Fuzzy (i, j)-semi open (closed) sets and fuzzy (i, j)-semi open (closed) maps in fuzzy bitopological spaces have been studied in [5].

In the present paper we introduce fuzzy (i, j)-regular open (closed) sets in fuzzy bitopological spaces. Significant results have been obtained. Further the concepts of fuzzy (i, j)-almost continuous maps and fuzzy (i, j)-weakly continuous maps, fuzzy (i, j)-semi regular and fuzzy (i, j)-regular spaces have also been introduced and studied in the present paper.

2. PRELIMINARIES

Let \( X \) be a nonempty set and let \( I \) stand for the closed unit interval \([0, 1]\). A fuzzy set \( \lambda \) of \( X \) is a mapping \( \lambda : X \rightarrow I \), where for any \( x \in X \), \( \lambda(x) \) denotes the degree of membership of element \( x \) in fuzzy set \( \lambda \). The null fuzzy set 0 and the whole fuzzy set 1 are the constant mappings from \( X \) to \( \{0\} \) and \( \{1\} \) respectively. The complement, union and intersection of fuzzy sets are defined as follows:

\[
\lambda'(x) = 1 - \lambda(x), \quad x \in X
\]

\[
\bigcup \lambda_{\alpha}(x) = \text{Sup} \{\lambda_{\alpha}(x) : \alpha \in \Lambda\}, \quad x \in X
\]

\[
\bigcap \lambda_{\alpha}(x) = \text{Inf} \{\lambda_{\alpha}(x) : \alpha \in \Lambda\}, \quad x \in X
\]

where \( \Lambda \) is any arbitrary index set.

Let \( f : X \rightarrow Y \) be a mapping and let \( \lambda \) be a fuzzy set in \( X \), then the image set \( f(\lambda) \) is a fuzzy set in \( Y \) defined as
for each \( y \in Y \). Further, if \( \mu \) is a fuzzy set of \( Y \), then \( f^{-1}(\mu) \) is a fuzzy set of \( X \) defined as
\[
f^{-1}(\mu)(x) = \mu(f(x))
\]
for each \( x \in X \).

A family \( \tau \) of fuzzy sets of \( X \) is called a fuzzy topology (see [2]) on \( X \) if it satisfies the following conditions:

i) The null fuzzy set 0 and whole fuzzy set 1 belong to \( \tau \).

ii) Any union of members of \( \tau \) is in \( \tau \).

iii) Any finite intersection of members of \( \tau \) is in \( \tau \).

The pair \((X, \tau)\) is called a fuzzy topological space. The members of \( \tau \) are called fuzzy open sets and their complements are called fuzzy closed sets. For a fuzzy set \( \lambda \) of \( X \), the interior \((\text{Int} \ \lambda)\) and the closure \((\text{Cl} \ \lambda)\) of \( \lambda \) are defined as
\[
\text{Int} \lambda = \text{Sup} \{ O : O \leq \lambda \text{ and } O \text{ is a fuzzy open set in } X \} \\
\text{Cl} \lambda = \text{Inf} \{ C : C \geq \lambda \text{ and } C \text{ is a fuzzy closed set in } X \}
\]

If \( X \) is a non-empty universal set, then a system \((X, \tau_1, \tau_2)\) consisting of set \( X \) and two fuzzy topologies \( \tau_1 \) and \( \tau_2 \) on \( X \) is called a fuzzy bitopological space (see [4]).

A fuzzy set \( \lambda \) of \((X, \tau_1, \tau_2)\) is called fuzzy \((i, j)\)-semi open set if there exists a fuzzy \( \tau_i \)-open set \( v \) such that \( v \leq \lambda \leq v - \text{Cl} \) \( v \) and \( \lambda \) is called fuzzy \((i, j)\)-semi closed set if there exists a fuzzy \( \tau_i \)-closed set \( \mu \) such that \( \tau_j - \text{Int} \ \mu \leq \lambda \leq \mu \) (see [5]). In this definition and in the rest of this paper we take \( i, j = 1, 2 \) & \( i \neq j \).

A fuzzy set \( \lambda \) is a fuzzy \((i, j)\)-semi open set iff \( \lambda \leq \tau_j - \text{Cl} (\tau_i - \text{Int} \ \lambda) \) and fuzzy \((i, j)\)-semi closed set iff \( \tau_j - \text{Int} (\tau_i - \text{Cl} \ \lambda) \leq \lambda \) (see [5]).

3. FUZZY \((i, j)\)-REGULAR OPEN (CLOSED) SETS

Definition 3.1: A fuzzy set \( \lambda \) of fuzzy bitopological space \((X, \tau_1, \tau_2)\) is called

i) Fuzzy \((i, j)\)-regular open set if \( \tau_j - \text{Int} (\tau_i - \text{Cl} \ \lambda) = \lambda \).

ii) Fuzzy \((i, j)\)-regular closed set if \( \tau_j - \text{Cl} (\tau_i - \text{Int} \ \lambda) = \lambda \).

Remark 3.1: (a) Every fuzzy \((i, j)\)-regular open set is a fuzzy \( \tau_j \)-open set, but converse need not be true.

(b) Every fuzzy \((i, j)\)-regular closed set is a fuzzy \( \tau_j \)-closed set, but converse need not be true.

We exemplify the remarks in the following:

Example 3.1: Let \( X = \{a, b\} \) and let \( A, B, C, D \) be fuzzy sets on \( X \) defined as follows
\[
A = \{(a, 0.7), (b, 0.5)\} \quad B = \{(a, 0.5), (b, 0.4)\} \\
C = \{(a, 0.3), (b, 0.4)\} \quad D = \{(a, 0.8), (b, 0.6)\}
\]

Consider \( \tau_1 = \{0, A, B, 1\} \) and \( \tau_2 = \{0, C, D, 1\} \) as two fuzzy topologies on \( X \). Then we find that fuzzy set \( C \) is fuzzy \((1, 2)\)-regular open set and it is a fuzzy \( \tau_2 \)-open set. Similarly fuzzy set \( E = \{(a, 0.7), (b, 0.6)\} = C \) is a fuzzy \((1, 2)\)-regular closed set and it is a fuzzy \( \tau_2 \)-closed set.
We observe that fuzzy set $D$ is a fuzzy $\tau_2$-open set, but it is not a fuzzy $(1, 2)$-regular open set because $\tau_1 - Cl D = 1$ and $\tau_2 - Int 1 = 1$. Thus $\tau_2 - Int (\tau_1 - Cl D) \neq D$.

**Theorem 3.1:** A fuzzy set $\lambda$ of a fuzzy bitopological space $(X, \tau_1, \tau_2)$ is a fuzzy $(i, j)$-regular open set if and only if $\lambda'$ is a fuzzy $(i, j)$-regular closed set.

**Proof:** Let $\lambda$ be a fuzzy $(i, j)$-regular open set of $X$, so that $\tau_j - Int (\tau_i - Cl \lambda) = \lambda$. It implies that $\lambda' = \tau_j - Cl (\tau_i - Int \lambda')$, which show that $\lambda'$ is a fuzzy $(i, j)$-regular closed set in $X$.

Conversely, let $\lambda'$ be a fuzzy $(i, j)$-regular closed set of fuzzy space $X$, so that $\tau_j - Cl (\tau_i - Int \lambda') = \lambda$. It implies $\tau_j - Int (\tau_i - Cl \lambda) = \lambda$. This proves that $\lambda$ is a fuzzy $(i, j)$-regular open set.

**Theorem 3.2:** (a) The intersection of two fuzzy $(i, j)$-regular open sets is a fuzzy $(i, j)$-regular open set.

(b) The union of two fuzzy $(i, j)$-regular closed sets is a fuzzy $(i, j)$-regular closed set.

**Proof:** (a) Let $(X, \tau_1, \tau_2)$ be a fuzzy bitopological space and let $\lambda$ and $\mu$ be two fuzzy $(i, j)$-regular open sets in it, so that

\[ \tau_j - Int (\tau_i - Cl \lambda) = \lambda \quad \text{and} \quad \tau_j - Int (\tau_i - Cl \mu) = \mu \] (3.2.1)

Thus $\lambda$ and $\mu$ are fuzzy $\tau_j$-open sets, hence $\lambda \cap \mu$ is also fuzzy $\tau_j$-open set. We see that

\[ \tau_j - Int (\lambda \cap \mu) = \lambda \cap \mu \leq \tau_i - Cl (\lambda \cap \mu) \]

Hence

\[ \lambda \cap \mu \leq \tau_j - Int (\tau_i - Cl (\lambda \cap \mu)) \] (3.2.2)

Further $\lambda \cap \mu \leq \lambda, \mu$. Therefore $\tau_i - Cl (\lambda \cap \mu) \leq \tau_i - Cl \lambda, \tau_i - Cl \mu$.

Hence

\[ \tau_j - Int (\tau_i - Cl (\lambda \cap \mu)) \leq \tau_i - Int (\tau_i - Cl \lambda), \tau_j - Int (\tau_i - Cl \mu) \]

In view of (3.2.1), we have

\[ \tau_j - Int (\tau_i - Cl (\lambda \cap \mu)) \leq \lambda \cap \mu \] (3.2.3)

Thus in view of (3.2.2) and (3.2.3), we have

\[ \tau_j - Int (\tau_i - Cl (\lambda \cap \mu)) = \lambda \cap \mu \]

Therefore $\lambda \cap \mu$ is fuzzy $(i, j)$-regular open set in $X$. Similarly we can prove (b).

**Remark 3.2:** Result (a) and (b) of Theorem 3.2 can be generalized to any finite number of fuzzy sets $\lambda_1, \lambda_2, ..., \lambda_n$.

**Theorem 3.3:** In a fuzzy bitopological space $(X, \tau_1, \tau_2)$,

(a) The $\tau_j$-closure of a fuzzy $\tau_i$-open set is a fuzzy $(i, j)$-regular closed set.

(b) The $\tau_j$-interior of a fuzzy $\tau_i$-closed set is a fuzzy $(i, j)$-regular open set.

**Proof:** We prove (a). Part (b) can be proved in a similar manner.

(a) Let $\lambda$ be a fuzzy $\tau_i$-open set of $(X, \tau_1, \tau_2)$. Consider the set $\theta = \tau_j - Cl \lambda$. We show that $\tau_j - Cl (\tau_i - Int \theta) = \theta$. Now $\lambda \leq \tau_j - Cl \lambda$ and $\tau_i - Int \lambda = \lambda$, so that

\[ \tau_i - Int (\tau_j - Cl \lambda) \leq \tau_j - Cl \lambda \]

Hence

\[ \tau_j - Cl (\tau_i - Int (\tau_j - Cl \lambda)) \leq \tau_j - Cl (\tau_j - Cl \lambda) \equiv \tau_j - Cl \lambda \]

Thus

\[ \tau_j - Cl (\tau_i - Int (\tau_j - Cl \lambda)) \leq \tau_j - Cl \lambda \] (3.3.1)

Now we know $\lambda \leq \tau_j - Cl \lambda$.

Then $\tau_i - Int \lambda \leq \tau_i - Int (\tau_j - Cl \lambda) \Rightarrow \lambda \leq \tau_i - Int (\tau_j - Cl \lambda)$.

Therefore

\[ \tau_j - Cl \lambda \leq \tau_j - Cl (\tau_i - Int (\tau_j - Cl \lambda)) \] (3.3.2)
The inequalities (3.3.1) and (3.3.2) imply that
\[ \tau_j - Cl(\tau_i - \text{Int}(\tau_j - Cl \lambda)) = \tau_j - Cl \lambda \]
Thus \( \tau_j \)-closure of a fuzzy \( \tau_i \)-open set is a fuzzy (i, j)-regular closed set.

4. FUZZY (i, j)-ALMOST CONTINUOUS MAPPING

Let \((X, \tau_1, \tau_2)\) and \((X^*, \tau_1^*, \tau_2^*)\) are two fuzzy bitopological spaces. We recall that a mapping \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) is said to be a

(i) fuzzy (i, j)-continuous map if maps \( f : (X, \tau_1) \rightarrow (X^*, \tau_1^*) \) and \( f : (X, \tau_2) \rightarrow (X^*, \tau_2^*) \) are fuzzy continuous maps.

(ii) fuzzy (i, j)-semi continuous map if pre-image of every fuzzy \( \tau_i^* \)-open set in \( X^* \) is a fuzzy (i, j)-semi open set in \( X \).

Now we proceed to define a fuzzy (i, j)-almost continuous map from one bitopological space to another bitopological space.

**Definition 4.1:** A map \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) is said to be a fuzzy (i, j)-almost continuous map if \( f^{-1}(\lambda) \) is a fuzzy \( \tau_j \)-open set in \( X \) for every fuzzy (i, j)-regular open set \( \lambda \) in \( X^* \).

**Example 4.1:** Let \( X = \{x, y\} \) and \( X^* = \{a, b\} \) and let \( A, B, C, A^*, B^*, C^* \) be the fuzzy sets defined as follows:
\[
A = \{(x,0.6), (y,0.5)\} \quad B = \{(x,0.3), (y,0.5)\} \quad C = \{(x,0.5), (y,0.4)\}
\[
A^* = \{(a,0.6), (b,0.5)\} \quad B^* = \{(a,0.3), (b,0.5)\} \quad C^* = \{(a,0.2), (b,0.4)\}
\]

Consider \( \tau_1 = \{0, A, C, 1\} \) and \( \tau_2 = \{0, B, 1\} \) as two fuzzy topologies on \( X \) and \( \tau_1^* = \{0, A^*, 1\} \) and \( \tau_2^* = \{0, B^*, C^*, 1\} \) as two fuzzy topologies on \( X^* \). Let \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) be a map defined as \( f(x) = a \) and \( f(y) = b \). Then we see 0, 1 and \( B^* \) are fuzzy (1,2)-regular open sets in \( X^* \). Also \( f^{-1}(0) \equiv 0 \), \( f^{-1}(1) \equiv 1 \) and \( f^{-1}(B^*) \equiv B \) are fuzzy \( \tau_2 \)-open sets in \( X \). Hence \( f \) is a fuzzy (1,2)-almost continuous map.

Similarly we observe that fuzzy sets 0, 1 and \( A^* \) are the only fuzzy (2, 1)-regular open sets in \( X^* \) and their pre-images are fuzzy \( \tau_1 \)-open sets in \( X \). Hence \( f \) is also a fuzzy (2, 1)-almost continuous map. Thus \( f \) is a fuzzy (i, j)-almost continuous map.

**Theorem 4.1:** Let \((X, \tau_1, \tau_2)\) and \((X^*, \tau_1^*, \tau_2^*)\) be two fuzzy bitopological spaces and let \( f : X \rightarrow X^* \) be a map, then following statements are equivalent:

(a) \( f \) is fuzzy (i, j)-almost continuous map.
(b) \( f^{-1}(\mu) \) is fuzzy \( \tau_j \)-closed set in \( X \) for each fuzzy (i, j)-regular closed set \( \mu \) in \( X^* \).
(c) \( f^{-1}(\lambda) \leq \tau_j - \text{Int}[f^{-1}(\tau_j^* - \text{Int}(\tau_j^* - Cl \lambda))] \) for every fuzzy \( \tau_j^* \)-open set \( \lambda \) of \( X^* \).
(d) For each fuzzy \( \tau_j^* \)-closed set \( \mu \) of \( X^* \), \( \tau_j - Cl[f^{-1}(\tau_j^* - Cl(\tau_j^* - \text{Int} \lambda))] \leq f^{-1}(\mu) \).

**Proof:** We prove the theorem in following steps:

(I) (a)\( \Rightarrow \) (b) : Suppose that the map \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) is a fuzzy (i, j)-almost continuous map. Let \( \mu \) be a fuzzy (i, j)-regular closed set in \( X^* \), then \( \mu' \) is fuzzy (i, j)-regular open set in \( X^* \). Therefore \( f^{-1}(\mu') \) is a fuzzy \( \tau_j \)-open set in \( X \). Hence \( (f^{-1}(\mu'))' = f^{-1}(\mu')' = f^{-1}(\mu) \) is a fuzzy \( \tau_j \)-closed set in \( X \).

(II) (b)\( \Rightarrow \) (a) : Using \( f^{-1}(\lambda')' = f^{-1}(\lambda') = f^{-1}(\lambda) \), where \( \lambda \) is a fuzzy (i, j)-regular open set in \( X^* \), it can be easily seen that (b)\( \Rightarrow \) (a).
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Let $f : X \to X^*$ be a fuzzy (i, j)-almost continuous map and let $\lambda$ be a fuzzy (i, j)-regular open set in $X^*$, so that $f^{-1}(\lambda)$ is a fuzzy $\tau_j$-open set in $X$. Now $\lambda \subseteq \tau_i^* - \text{Cl} \lambda$. Then

$$\tau_j^* - \text{Int} \lambda \equiv \lambda \subseteq \tau_j^* - \text{Int}(\tau_i^* - \text{Cl} \lambda).$$

Hence

$$f^{-1}(\lambda) \subseteq f^{-1}\{\tau_j^* - \text{Int}(\tau_i^* - \text{Cl} \lambda)\} \quad (4.1.1)$$

Since $f$ is fuzzy (i, j)-almost continuous map, therefore $f^{-1}\{\tau_j^* - \text{Int}(\tau_i^* - \text{Cl} \lambda)\}$ is a fuzzy $\tau_j$-open set in $X$ in view of Theorem 3.3(b). So that

$$\tau_j - \text{Int}[f^{-1}\{\tau_j^* - \text{Int}(\tau_i^* - \text{Cl} \lambda)\}] = f^{-1}\{\tau_j^* - \text{Int}(\tau_i^* - \text{Cl} \lambda)\}$$

Hence we get $f^{-1}(\lambda) \subseteq \tau_j - \text{Int}[f^{-1}\{\tau_j^* - \text{Int}(\tau_i^* - \text{Cl} \lambda)\}]$ in view of (4.1.1).

IV (c)$\Rightarrow$(a): It can be proved easily.

(V) (b)$\Rightarrow$(d): Let $\mu$ be a fuzzy $\tau_j^*$-closed set in $X^*$, so that $\mu = \tau_j^* - \text{Cl} \mu$. Also $\tau_i^* - \text{Int} \mu \subseteq \tau_j^* - \text{Cl} \mu = \mu$. Hence $f^{-1}\{\tau_j^* - \text{Cl}(\tau_i^* - \text{Int} \mu)\} \subseteq f^{-1}(\mu)$. Since $f$ satisfies the property that $f^{-1}(\mu)$ is fuzzy $\tau_j$ -closed set in $X$, for each fuzzy (i, j)-regular closed set $\mu$ in $X^*$, we conclude that $f^{-1}\{\tau_j^* - \text{Cl}(\tau_i^* - \text{Int} \mu)\}$ is a fuzzy $\tau_j$-closed set in $X$. Hence

$$\tau_j - \text{Cl}[f^{-1}\{\tau_j^* - \text{Cl}(\tau_i^* - \text{Int} \mu)\}] = f^{-1}\{\tau_j^* - \text{Cl}(\tau_i^* - \text{Int} \mu)\} \subseteq f^{-1}(\mu).$$

(VI) (d)$\Rightarrow$(b): It can be proved easily.

This completes the proof of the Theorem.

**Theorem 4.2**: If a map $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is a fuzzy (i, j)-continuous map, then it is a fuzzy (i, j)-almost continuous map.

**Proof**: Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ be a fuzzy (i, j)-continuous map so that maps $f : (X, \tau_1) \to (X^*, \tau_1^*)$ and $f : (X, \tau_2) \to (X^*, \tau_2^*)$ are continuous. Now if $\theta$ is fuzzy (i, j)-regular open set in $X^*$, then $\theta$ is a fuzzy $\tau_j^*$-open set in $X^*$. Since $f$ is a fuzzy (i, j)-continuous map, therefore $f^{-1}(\theta)$ is a fuzzy $\tau_j$-open set in $X$. Hence map $f : X \to X^*$ is fuzzy (i, j)-almost continuous map.

**Remark 4.1**: The converse of Theorem 4.2 may not be true i.e. every fuzzy (i, j)-almost continuous map is not necessarily a fuzzy (i, j)-continuous map.

**Example 4.2**: Considering Example 4.1, we see that $f$ is a fuzzy (1, 2)-almost continuous map, but it is not a fuzzy (1, 2)-continuous map and also a fuzzy (2, 1)-almost continuous map. But we observe that the mapping $f : (X, \tau_2) \to (X^*, \tau_2^*)$ is not a continuous map because $f^{-1}(C^*) \not\subseteq \tau_2^*$ where $C^*$ is a fuzzy $\tau_2^*$-open set in $X^*$ and thus the map $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is not fuzzy (i, j)-continuous map.

**Theorem 4.3**: Fuzzy (i, j)-semi continuity and fuzzy (i, j)-almost continuity are independent notions.

Following two examples justify the statement of the theorem.

**Example 4.3**: Referring to Example 4.1, we see that $f$ is a fuzzy (1, 2)-almost continuous map, but it is not a fuzzy (1, 2)-semi continuous map because $f^{-1}(A^*) \equiv A$ is not a fuzzy (1, 2)-semi open set in $X$, where $A^*$ is a fuzzy $\tau_1^*$-open set in $X^*$.

Similarly $f$ is not a fuzzy (2, 1)-semi continuous map because $f^{-1}(B^*) \equiv B$ is not a fuzzy (2, 1)-semi open set in $X$, $B^*$ being $\tau_2^*$-open set in $X^*$.

**Example 4.4**: Let $X = \{x, y\}$ and $X^* = \{a, b\}$ and let $A, B, A^*, B^*$ be the fuzzy sets defined as follows

$A = \{(x, 0.4), (y, 0.5)\}$ \quad $B = \{(x, 0.2), (y, 0.4)\}$

$A^* = \{(a, 0.6), (b, 0.5)\}$ \quad $B^* = \{(a, 0.3), (b, 0.5)\}$
Consider \( \tau_1 = \{0, A, 1\} \) and \( \tau_2 = \{0, B, 1\} \) as two fuzzy topologies on \( X \) and \( \tau_1^* = \{0, A^*, 1\} \) and \( \tau_2^* = \{0, B^*, 1\} \) be two fuzzy topologies on \( X^* \). Let \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) be a map defined as \( f(x) = a \) and \( f(y) = b \). We observe that map \( f \) is a fuzzy (i, j)-semi continuous map, but it is not a fuzzy (i, j)-almost continuous map.

We see that the map \( f \) is a fuzzy (1, 2)-semi continuous map because \( f^{-1}(0) = 0 \), \( f^{-1}(1) = 1 \) and \( f^{-1}(A^*) = \{(x, 0.6), (b, 0.5)\} \) are fuzzy (1, 2)-semi open sets in \( X \) for \( \tau_1^* \)-open sets 0, 1 and \( A^* \)in \( X^* \). Similarly we observe that the map \( f \) is a fuzzy (2, 1)-semi continuous map.

But we observe that \( f \) is not a fuzzy (2, 1)-almost continuous map because \( f^{-1}(A^*) \) is not a fuzzy \( \tau_1 \)-open set in \( X \) for fuzzy (2, 1)-regular open set \( A^* \)in \( X^* \). Also \( f \) is not a fuzzy (1, 2)-almost continuous map because \( f^{-1}(B^*) \) is not a fuzzy \( \tau_2 \)-open set in \( X \) for fuzzy (1, 2)-regular open set \( B^* \)in \( X^* \). Thus the map \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) is not a fuzzy (i, j)-almost continuous map.

5. **Fuzzy (i, j)-SEMI REGULAR & FUZZY (i, j)-REGULAR SPACES**

**Definition 5.1: Fuzzy (i, j)-semi regular space**: A fuzzy bitopological space \((X, \tau_1, \tau_2)\) is called a fuzzy (i, j)-semi regular space if and only if the collection of all fuzzy (i, j)-regular open sets of \( X \) forms a base for fuzzy topology \( \tau_j \) of \( X \) with \( i, j = 1, 2 \).

**Example 5.1**: Let \( X = \{a, b\} \) and let \( A, B, C, D, E, F, G, H \) be fuzzy sets on \( X \) defined as follows:

\[
\begin{align*}
A &= \{(a, 0.4), (b, 0.7)\} \\
B &= \{(a, 0.6), (b, 0.3)\} \\
C &= A \cup B = \{(a, 0.6), (b, 0.7)\} \\
D &= A \cap B = \{(a, 0.4), (b, 0.3)\} \\
E &= \{(a, 0.5), (b, 0.2)\} \\
F &= \{(a, 0.3), (b, 0.6)\} \\
G &= E \cup F = \{(a, 0.5), (b, 0.6)\} \\
H &= E \cap F = \{(a, 0.3), (b, 0.2)\}
\end{align*}
\]

Consider \( \tau_1 = \{0, A, B, C, D, 1\} \) and \( \tau_2 = \{0, E, F, G, H, 1\} \) as two fuzzy topologies on \( X \). Since fuzzy sets \( 0, 1, E, F, G \) and \( H \) are the only fuzzy (1, 2)-regular open sets of \( X \) and we can write each of fuzzy \( \tau_2 \)-open sets as union of some of these six fuzzy (1, 2)-regular open sets. Hence fuzzy bitopological space \((X, \tau_1, \tau_2)\) is a fuzzy (1, 2)-semi regular space.

Similarly fuzzy sets \( 0, 1, A, B, C \) and \( D \) are the only fuzzy (2, 1)-regular open sets of \( X \) and we can write each of fuzzy \( \tau_1 \)-open sets as union of some of these six fuzzy (2, 1)-regular open sets, which form a base for fuzzy topology \( \tau_1 \) of \( X \). Hence fuzzy bitopological space \((X, \tau_1, \tau_2)\) is a fuzzy (2, 1)-regular semi regular space.

Thus fuzzy bitopological space \((X, \tau_1, \tau_2)\) is a fuzzy (i, j)-semi regular space.

**Theorem 5.1**: Let \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) be a map from fuzzy bitopological space \((X, \tau_1, \tau_2)\) to a fuzzy (i, j)-semi regular space \((X^*, \tau_1^*, \tau_2^*)\). Then the map \( f \) is fuzzy (i, j)-almost continuous if and only if \( f \) is fuzzy (i, j)-continuous map.

**Proof**: We know (by Theorem 4.2) that any fuzzy (i, j)-continuous map from one fuzzy bitopological space to another is a fuzzy (i, j)-almost continuous map. Therefore to prove the theorem, it is sufficient to show that if \( (X^*, \tau_1^*, \tau_2^*) \) is fuzzy (i, j)-semi regular space and map \( f \) is a fuzzy (i, j)-almost continuous map, then it is a fuzzy (i, j)-continuous map. Suppose that \( (X^*, \tau_1^*, \tau_2^*) \) is fuzzy (i, j)-semi regular space and the map \( f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*) \) is fuzzy (i, j)-almost continuous map. Let \( \lambda \) be a fuzzy \( \tau_1^* \)-open set in \( X^* \). Then \( \lambda \) is the union of a collection of fuzzy (i, j)-regular open sets \( \{\lambda_\alpha\}_{\alpha \in \Lambda} \) in \( X^* \), where \( \Lambda \) is an arbitrary index set. Thus \( \lambda = \bigcup_{\alpha \in \Lambda} \lambda_\alpha \). Since each \( \lambda_\alpha \) is fuzzy (i, j)-regular open set, we have \( \lambda_\alpha = \tau_j^* - \text{int}(\tau_i^* - C \text{int} \lambda_\alpha) \), \( \forall \alpha \in \Lambda \).
Therefore \( f^{-1}(\lambda) = f^{-1}(U_\alpha \lambda_\alpha) = \bigcup_\alpha f^{-1}(\lambda_\alpha) \). Then
\[
 f^{-1}(\lambda) \subseteq \bigcup_\alpha \tau_j - \text{Int}[f^{-1}[\tau_j^* - \text{Int}(\tau_j^* \cap \lambda_\alpha)]] = \bigcup_\alpha \tau_j - \text{Int}[f^{-1}(\lambda_\alpha)] \quad \text{in view of Theorem 4.1(c). This implies} \quad f^{-1}(\lambda) \subseteq \tau_j - \text{Int}[\bigcup_\alpha f^{-1}(\lambda_\alpha)] = \tau_j - \text{Int}[f^{-1}(\lambda)].
\]
Thus \( f^{-1}(\lambda) \leq \tau_j - \text{Int}[f^{-1}(\lambda)] \). Therefore, we clearly have
\[
 \tau_j - \text{Int}[f^{-1}(\lambda)] \leq f^{-1}(\lambda) \leq \tau_j - \text{Int}[f^{-1}(\lambda)]
\]
which shows that \( f^{-1}(\lambda) \) is a fuzzy \( \tau_j \)-open set in \( X \). Thus when \( X^* \) is a fuzzy (i, j)-semi regular space and \( \lambda \) is a fuzzy \( \tau_j^* \)-open set in \( X^* \), then \( f^{-1}(\lambda) \) is a fuzzy \( \tau_j \)-open set in \( X \) for \( j = 1, 2 \).

Therefore \( f \) is a fuzzy (i, j)-continuous map from \((X, \tau_1, \tau_2)\) to \((X^*, \tau_1^*, \tau_2^*)\).

**Definition 5.2: Fuzzy (i, j)-regular space** : A fuzzy bitopological space \((X, \tau_1, \tau_2)\) is called a fuzzy (i, j)-regular bitopological space if and only if each fuzzy \( \tau_j \)-open set \( \lambda \) is a union of fuzzy \( \tau_j \)-open sets \( \lambda_\alpha \) of \( X \) such that \( \tau_i - \text{Cl} \lambda_\alpha \leq \lambda \).

**Example 5.2:** Let \( \lambda \) and \( \mu \) be two fuzzy sets of \( X \) defined as follows
\[
\lambda(x) = x \quad \text{and} \quad \mu(x) = 1 - x, \forall x \in X.
\]
Consider fuzzy topologies \( \tau_1 = \{0, \lambda, 1\} \) and \( \tau_2 = \{0, \mu, 1\} \) on \( X \). It is clear that 0, \( \mu \) and 1 are fuzzy \( \tau_2 \)-open sets of \( X \). Then we see that
\[
0 = 0 \cup 0, \quad \text{and} \quad \tau_1 - \text{Cl} 0 = 0 \leq 0
\]
\[
\mu = 0 \cup \mu, \quad \text{and} \quad (i) \quad \tau_1 - \text{Cl} 0 = 0 \leq \mu \quad \text{and}
\]
\[
1 = 0 \cup \mu \cup 1, \quad \text{and} \quad (i) \quad \tau_1 - \text{Cl} 0 = 0 \leq 1, \quad (ii) \quad \tau_1 - \text{Cl} \mu = \lambda' = \mu
\]

Thus conditions of a fuzzy (1, 2)-regular space are satisfied. Hence fuzzy bitopological space \((X, \tau_1, \tau_2)\) is a fuzzy (1, 2)-regular space.

Similarly we observe that each fuzzy \( \tau_1 \)-open sets 0, \( \lambda \) and 1 are the union of fuzzy \( \tau_1 \)-open sets, which satisfy the condition of the definition. Therefore fuzzy bitopological space \((X, \tau_1, \tau_2)\) is a fuzzy (2, 1)-regular space also. Hence \((X, \tau_1, \tau_2)\) is a fuzzy (i, j)-regular space.

**Theorem 5.2:** A fuzzy (i, j)-regular bitopological space is also a fuzzy (i, j)-semi regular bitopological space.

**Proof:** Let \((X, \tau_1, \tau_2)\) be a fuzzy (i, j)-regular bitopological space. Let \( \lambda \) be a fuzzy \( \tau_j \)-open set in \( X \). Suppose \( \lambda \) is the union of family \( \{\lambda_\alpha\}_{\alpha \in \Lambda} \) of fuzzy \( \tau_j \)-open sets \( \lambda_\alpha \) such that \( \tau_i - \text{Cl} \lambda_\alpha \leq \lambda, \forall \alpha \in \Lambda \), \( \Lambda \) being an arbitrary index set. Then \( \lambda = \bigcup_\alpha \lambda_\alpha, \alpha \in \Lambda \).

Now for each \( \alpha \in \Lambda \), we have \( \lambda_\alpha \leq \tau_i - \text{Cl} \lambda_\alpha \leq \lambda \). Then
\[
\tau_j - \text{Int} \lambda_\alpha \equiv \lambda_\alpha \leq \tau_j - \text{Int}(\tau_i - \text{Cl} \lambda_\alpha) \leq \tau_j - \text{Int} \lambda \equiv \lambda
\]
Thus \( \lambda_\alpha \leq \tau_j - \text{Int}(\tau_i - \text{Cl} \lambda_\alpha) \leq \lambda \). Hence \( \bigcup_\alpha \lambda_\alpha \leq \bigcup_\alpha \tau_j - \text{Int}(\tau_i - \text{Cl} \lambda_\alpha) \leq \lambda \).

It implies \( \lambda = \bigcup_\alpha \lambda_\alpha = \bigcup_\alpha \tau_j - \text{Int}(\tau_i - \text{Cl} \lambda_\alpha) \leq \lambda \).

Thus
\[
\lambda = \bigcup_\alpha \tau_j - \text{Int}(\tau_i - \text{Cl} \lambda_\alpha) \quad (5.2.1)
\]

Since we know that the fuzzy sets \( \tau_j - \text{Int}(\tau_i - \text{Cl} \lambda_\alpha) \) is a fuzzy (i, j)-regular open set in \( X \). Hence (5.2.1) indicates that \( \lambda \) is a union of fuzzy (i, j)-regular open sets of \( X \). Therefore \( X \) is a fuzzy (i, j)-semi regular bitopological space.
Remark 5.1: A fuzzy (i, j)-semi regular space is not necessarily a fuzzy (i, j)-regular space. This is shown in the following example.

Example 5.3: Consider fuzzy bitopological space $(X, \tau_1, \tau_2)$ of Example 5.1. It is a fuzzy (1, 2)-semi regular space. We note that $\tau_2$-open fuzzy set $G$ is the union of fuzzy $\tau_2$-open sets $E$ and $F$, but
\[
\tau_1 - \text{Cl} E = A' \notin G \quad \text{and} \quad \tau_1 - \text{Cl} F = B' \notin G
\]
Hence fuzzy (1, 2)-semi regular bitopological space $(X, \tau_1, \tau_2)$ is not a fuzzy (1, 2)-regular space.

6. FUZZY (i, j)-WEAKLY CONTINUOUS MAP

Definition 6.1: Let $(X, \tau_1, \tau_2)$ and $(X', \tau_1', \tau_2')$ be two fuzzy bitopological spaces. A map $f : (X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ is called fuzzy (i, j)-weakly continuous map if for each fuzzy $\tau_j^*$-open set $\lambda$ in $X^*$
\[
f^{-1}(\lambda) \subseteq \tau_j - \text{Int} \{f^{-1}(\tau_i^* - \text{Cl} \lambda)\}
\]

Example 6.1: Referring to Example 4.1, we see that fuzzy sets $0, 1, B^*$ and $C^*$ are fuzzy $\tau_2^*$-open sets in $X^*$. We note that $f^{-1}(B^*) = B$ and $f^{-1}(C^*) = \{x, y, 0.4\} \subseteq B$. We observe that
\[
\tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} 0)\} = 0, \quad \text{so that} \quad f^{-1}(0) \equiv 0 \subseteq \tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} 0)\}
\]
\[
\tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} 1)\} = 1, \quad \text{so that} \quad f^{-1}(1) \equiv 1 \subseteq \tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} 1)\}
\]
\[
\tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} B^*)\} = B, \quad \text{so that} \quad f^{-1}(B^*) = B \subseteq \tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} B^*)\}
\]
\[
\tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} C^*)\} = B, \quad \text{so that} \quad f^{-1}(C^*) \subseteq \tau_2 - \text{Int} \{f^{-1}(\tau_1^* - \text{Cl} C^*)\}
\]
In view of (a), (b), (c), (d), we conclude that the mapping $f$ is a fuzzy (1, 2)-weakly continuous map.

Similarly we observe that the mapping $f$ is a fuzzy (2, 1)-weakly continuous map.

Theorem 6.1: A fuzzy (i, j)-continuous map from one bitopological space to another is a fuzzy (i, j)-weakly continuous map.

Proof: Let $f : (X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ be a fuzzy (i, j)-continuous map. Let $\lambda$ be any $\tau_j^*$-open set in $X^*$, then $f^{-1}(\lambda)$ is a fuzzy $\tau_j$-open set in $X$. We know $\lambda \subseteq \tau_i^* - \text{Cl} \lambda$. Therefore $f^{-1}(\lambda) \subseteq f^{-1}(\tau_i^* - \text{Cl} \lambda)$. Since $f^{-1}(\lambda)$ is a fuzzy $\tau_j$-open set in $X$, we have
\[
\tau_j - \text{Int} \{f^{-1}(\lambda)\} = f^{-1}(\lambda) \subseteq \tau_j - \text{Int} \{f^{-1}(\tau_i^* - \text{Cl} \lambda)\}
\]
Therefore
\[
f^{-1}(\lambda) \subseteq \tau_j - \text{Int} \{f^{-1}(\tau_i^* - \text{Cl} \lambda)\}, \quad \forall \lambda \in \tau_j^*
\]
This shows that $f$ is a fuzzy (i, j)-weakly continuous map.

Remark 6.1: The converse of above Theorem 6.1 need not be true i.e. every fuzzy (i, j)-weakly continuous map is not necessarily a fuzzy (i, j)-continuous map. We have the following:

Example 6.2: Considering Example 6.1 we have shown that the map $f$ is a fuzzy (1, 2)-weakly continuous and also a fuzzy (2, 1)-weakly continuous map, whereas in Examples 4.2, we observe that the map $f$ is not a fuzzy (i, j)-continuous map.

Theorem 6.2: A fuzzy (i, j)-almost continuous map is a fuzzy (i, j)-weakly continuous map.

Proof: Let $f : (X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ be a fuzzy (i, j)-almost continuous map. Let $\lambda$ be any $\tau_j^*$-open set in $X^*$. Then we have
\[
f^{-1}(\lambda) \subseteq \tau_j - \text{Int} \{f^{-1}(\tau_i^* - \text{Int} (\tau_i^* - \text{Cl} \lambda))\}
\]
(i, j)-Almost Continuity and (i, j)-Weakly Continuity in Fuzzy Bitopological Spaces

in view of Theorem 4.1(c). Further, we clearly have \( \tau_j^s - \text{Int}(\tau_i^s - \text{Cl} \lambda) \leq \tau_i^s - \text{Cl} \lambda \). Therefore \( f^{-1}[\tau_j^s - \text{Int}(\tau_i^s - \text{Cl} \lambda)] \leq f^{-1}(\tau_i^s - \text{Cl} \lambda) \). Hence

\[
\tau_j - \text{Int}[f^{-1}(\tau_j^s - \text{Int}(\tau_i^s - \text{Cl} \lambda))] \leq \tau_j - \text{Int}[f^{-1}(\tau_i^s - \text{Cl} \lambda)]
\]  

(6.2.2)

Therefore in view of (6.2.1) & (6.2.2), we conclude that

\[
f^{-1}(\lambda) \leq \tau_j - \text{Int} \{ f^{-1}(\tau_i^s - \text{Cl} \lambda) \}, \quad \forall \lambda \in \tau_j^s
\]

Thus \( f \) is a fuzzy (i, j)-weakly continuous map.

**Remark 6.2:** A fuzzy (i, j)-weakly continuous map may not be a fuzzy (i, j)-semi continuous map.

**Example 6.3:** It is clear from Examples 6.1 that \( f \) is a fuzzy (1, 2)-weakly continuous map (and fuzzy (2, 1)-weakly continuous map), but in Example 4.3 we observe that the map \( f \) is not a fuzzy (1, 2)-semi continuous map (and fuzzy (2, 1)-semi continuous map).

**Remark 6.3:** A fuzzy (i, j)-semi continuous map may not be a fuzzy (i, j)-weakly continuous map.

**Example 6.4:** Referring Example 4.4, we see that map \( f \) is a fuzzy (i, j)-semi continuous map. But it is not a fuzzy (1, 2)-weakly continuous map because for any fuzzy \( \tau_2^s \)-open set \( B^s \) in \( X^s \), we observe that \( \tau_2 - \text{Int} \{ f^{-1}(\tau_i^s - \text{Cl} B^s) \} = B \) and \( f^{-1}(B^s) = [(x, 0.3), (y, 0.5)] \), so that \( f^{-1}(B^s) \geq B \).

Similarly \( f \) is not a fuzzy (2, 1)-weakly continuous map because for fuzzy \( \tau_1^s \)-open set \( A^s \) in \( X^s \), we see that \( \tau_1 - \text{Int} \{ f^{-1}(\tau_2^s - \text{Cl} A^s) \} = A \) and \( f^{-1}(A^s) = [(x, 0.6), (y, 0.5)] \) and hence \( f^{-1}(A^s) \geq A \).

Thus we have the following Theorem.

**Theorem 6.3:** In case of fuzzy bitopological spaces the notions of fuzzy (i, j)-weakly continuity and fuzzy (i, j)-semi continuity are independent of each other.

**Theorem 6.4:** Let \( f : (X, \tau_i, \tau_j) \rightarrow (X^s, \tau_i^s, \tau_j^s) \) be a mapping from a fuzzy bitopological space \((X, \tau_i, \tau_j)\) to a fuzzy (i, j)-regular bitopological space \((X^s, \tau_i^s, \tau_j^s)\). Then \( f \) is a fuzzy (i, j)-weakly continuous map if and only if \( f \) is fuzzy (i, j)-continuous map.

**Proof:** To prove the Theorem, in view of Theorem 6.1, it is sufficient to show that if mapping \( f : (X, \tau_i, \tau_j) \rightarrow (X^s, \tau_i^s, \tau_j^s) \) is a fuzzy (i, j)-weakly continuous and if \((X^s, \tau_i, \tau_j)\) is a fuzzy (i, j)-regular bitopological space, then map \( f \) is a fuzzy (i, j)-continuous map. Let \( \lambda \) be a fuzzy \( \tau_j^s \)-open set in \( X^s \). Since \( X^s \) is a fuzzy (i, j)-regular space, we have

\[
\lambda = \bigcup_{\alpha} \lambda_{\alpha}, \quad \alpha \in \Lambda, \text{ for some index set } \Lambda,
\]

and for each \( \alpha \in \Lambda, \lambda_{\alpha} \) is a fuzzy \( \tau_j^s \)-open sets in \( X^s \) such that \( \tau_i^s - \text{Cl} \lambda_{\alpha} \leq \lambda \)

(6.4.1)

Since \( f \) is a fuzzy (i, j)-weakly continuous map and \( \lambda_{\alpha} \) are fuzzy \( \tau_j^s \)-open sets, we have

\[
f^{-1}(\lambda_{\alpha}) \leq \tau_j - \text{Int} \{ f^{-1}(\tau_i^s - \text{Cl} \lambda_{\alpha}) \}, \quad \forall \alpha \in \Lambda
\]

(6.4.3)

Now in view of (6.4.1) we have \( f^{-1}(\lambda) = f^{-1}(\bigcup_{\alpha} \lambda_{\alpha}) = \bigcup_{\alpha} f^{-1}(\lambda_{\alpha}) \).

Therefore \( f^{-1}(\lambda) \leq \bigcup_{\alpha} \tau_j - \text{Int} \{ f^{-1}(\tau_i^s - \text{Cl} \lambda_{\alpha}) \} \), in view of (6.4.3).

This implies that \( f^{-1}(\lambda) \leq \bigcup_{\alpha} \tau_j - \text{Int} \{ f^{-1}(\lambda) \} = \tau_j - \text{Int} \{ f^{-1}(\lambda) \} \).

Thus \( f^{-1}(\lambda) \leq \tau_j - \text{Int} \{ f^{-1}(\lambda) \} \).

Hence we conclude that \( f^{-1}(\lambda) = \tau_j - \text{Int} \{ f^{-1}(\lambda) \} \). Thus \( f^{-1}(\lambda) \) is a fuzzy \( \tau_j^s \)-open set of \( X \) for fuzzy \( \tau_j^s \)-open set \( \lambda \) in \( X^s \) and this happens for each \( j = 1, 2 \). Therefore \( f \) is a fuzzy (i, j)-continuous map.
7. CONCLUSION

In [5], fuzzy $(i, j)$-semi open sets and fuzzy $(i, j)$-semi continuity in fuzzy bitopological spaces have been studied. In the present paper, we introduced the concepts of fuzzy regular open (closed) sets and studied fuzzy $(i, j)$-almost continuity and $(i, j)$-weakly continuity in fuzzy bitopological spaces. We have also studied fuzzy $(i, j)$-semi regular and fuzzy $(i, j)$-regular spaces in fuzzy bitopological spaces.

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