On \( \theta \)-Semigeneralized Pre Closed Sets in Topological Spaces

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Abstract: This paper introduces new class of sets called \( \theta \)-semigeneralized pre closed set in topological spaces. Basic properties of this new generalized closed sets are analysed.

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1. INTRODUCTION

General topology is important in many fields of applied sciences as well as in all branches of mathematics. The concept of generalized closed sets introduced by Levine[13] plays important role in general topology. This notion has been extensively studied in recent years by many topologists. Bhattacharyya and Lahiri [2] continued the work of Levine and offered another notion analogous to Levine’s g-closed sets called semi-generalized closed set (briefly sg-closed) by replacing the closure operator in Levine’s g-closed set by semi-closure operator and by replacing its open super set by semi-open super set. Recently, Dontchev and Maki [9] gave another new generalization of Levin’s g-closed set by utilizing \( \theta \)-closure operator called \( \theta \)-g-closed set. The concept of \( \theta \)-g-closed set was applied to the digital line. In 2003, Caldas and Jafari defined \( \theta \)-semigeneralized closed set using semi-\( \theta \)-closure operator.

In section three, we introduce a new form of generalized closed set called \( \theta \)-semigeneralized pre closed set (briefly, \( \theta \)-sgp-closed set) by utilizing pre-\( \theta \)-closure operator. We investigate its relation to \( \theta \)-g-closed sets, \( \theta \)-sg-closed sets and other generalized closed sets. We have proved that the class of \( \theta \)-sg-closed sets and the class of \( \theta \)-sgp-closed sets are independent.

2. PRELIMINARIES

Throughout this paper \((X, \tau)\) and \((Y, \sigma)\) (or simply \(X\) and \(Y\)) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If \(A\) is any subset of space \(X\), then \(\text{Cl}(A)\) and \(\text{Int}(A)\) denote the closure of \(A\) and the interior of \(A\) in \(X\) respectively.

The following definitions are useful in the sequel.

**Definition 2.1:** A subset \(A\) of space \(X\) is called

(i) a semi-open set [12] if \(A \subseteq \text{Cl}(\text{Int}(A))\).

(ii) a semi-closed set [5] if \(\text{Int}(\text{Cl}(A)) \subseteq A\).

(iii) a pre-open set[15] if \(A \subseteq \text{Int}(\text{Cl}(A))\).

(iv) a pre-closed set[15] if \(\text{Cl}(\text{Int}(A)) \subseteq A\).

(v) an \(\alpha\)-closed set[16] if \(\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A\).

(vi) a regular open set[21](resp. a regular closed set[21]) if \(A = \text{Int}(\text{Cl}(A))\)(resp. \(A = \text{Cl}(\text{Int}(A))\)).
Definition 2.2: A subset A of a topological space X is called

(i) a generalized-closed (briefly g-closed) set[13] if Cl(A) ⊆ U and U is open in X.

(ii) a semi-generalized closed set (briefly sg-closed)[2] if sCl(A) ⊆ U and U is semi-open in X. The complement of a sg-closed set is called a sg-open set.

(iii) a semi-generalized pre closed set (briefly sgp-closed)[17] if pCl(A) ⊆ U whenever A ⊆ U and U is semi-open in X.

(iv) a generalized preregular closed set(briefly gpr-closed)[11] if pCl(A) ⊆ U whenever A ⊆ U and U is regular open in X.

(v) an α-generalized semi-closed set(briefly ags-closed)[20] if αCl(A) ⊆ U whenever A ⊆ U and U is semi-open in X.

(vi) a generalized preclosed set(briefly gp-closed)[14] if pCl(A) ⊆ U whenever A ⊆ U and U is open in X.

(vii) a generalized semi-preclosed set(briefly gsp-closed)[8] if spCl(A) ⊆ U whenever A ⊆ U and U is open in X.

(viii) θ-generalized closed set(briefly θ-g-closed)[9] if Cl_θ(A) ⊆ U whenever A ⊆ U and U is open in X.

(ix) θ-generalized semi-closed set(briefly θ-gs-closed)[18] if sCl_θ(A) ⊆ U whenever A ⊆ U and U is open in X.

(x) θ-semigeneneralized closed set(briefly θ-sg-closed)[4] if sCl_θ(A) ⊆ U whenever A ⊆ U and U is semi-open in X.

Definition 2.3: The semi-closure [5] of a subset A of X is the intersection of all semi-closed sets that contain A and is denoted by sCl(A).

Definition 2.4: The pre-closure [6] of a subset A of X is the intersection of all pre-closed sets that contain A and is denoted by pCl(A).

Definition 2.5: The θ-closure [22] of a set A is denoted by Cl_θ(A) and is defined by Cl_θ(A) = {x ∈ X : Cl(U) ∩ A ≠ Ø, U ∈ τ, x ∈ U} and a set A is θ-closed if and only if A = Cl_θ(A).

Definition 2.6: A point x ∈ X is called a semi-θ-cluster point of A [7] if sCl(U) ∩ A ≠ Ø, for each semi-open set U containing x.

Definition 2.7: A point x ∈ X is called a pre-θ-cluster point of A[19] if pCl(U) ∩ A ≠ Ø, for each pre-open set U containing x.

Definition 2.8: The semi-θ-closure [7] denoted by sCl_θ(A), is the set of all semi-θ-cluster points of A. A subset A is called semi-θ-closed set [7] if A = sCl_θ(A). The complement of semi-θ-closed set is semi-θ-open set.

Definition 2.9: The pre-θ-closure denoted by pCl_θ(A), is the set of all pre-θ-cluster points of A. A subset A is called pre-θ-closed set [19] if A = pCl_θ(A). The complement of pre-θ-closed set is pre-θ-open set.

Definition 2.10: The set \{x ∈ X | sCl(U) ⊆ A for some U ∈ SO(X, x)\} is called the semi-θ-interior of A and is denoted by sInt_θ(A). A subset A is called semi-θ-open[10] if A = sInt_θ(A).

Definition 2.11: A topological space X is a pre-θ-R_0 space[1] if every pre-θ-open set contains pre-θ-closure of each of its singletons.

Definition 2.12: Let A be subset of a topological space X. The pre-θ-kernal[1] of A ⊆ X, denoted by pKer_θ(A), is defined to be the set ∩\{O : O ∈ P_0O(X, τ) and A ⊆ O\}.
On 0-Semigeneralized Pre Closed Sets in Topological Spaces

Lemma 2.13[3]: For any subset A of a topological space X, $pCl(A) \subseteq pCl_0(A)$.

3. 0-SEMIGENERALIZED PRE CLOSED SETS

We introduce the following definition.

Definition 3.1: A subset $A$ of a topological space $X$ is called 0-Semigeneralized pre closed set (briefly, 0-sgp-closed set) if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$.

The complement of 0-Semigeneralized pre closed set is called 0-Semigeneralized pre open set (briefly, 0-sgp-open).

Remark 3.2: The concept of 0-sgp-closed sets and closed sets are independent of each other as seen from the following examples.

Example 3.3: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}\}$. Then the subset $A = \{a, c\}$ is 0-sgp-closed set but it is not closed set in $X$.

Example 3.4: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Then the subset $A = \{b, c\}$ is closed set but it is not 0-sgp-closed set in $X$.

Theorem 3.5: Every pre-0-closed set is 0-sgp-closed set but not conversely.

Proof: Let $A \subseteq U$ be pre-0-closed. Then $A = pCl(A)$. Let $A \subseteq U$ and $U$ is semi-open in $X$. It follows that $pCl(A) \subseteq U$. This means that $A$ is 0-sgp-closed set.

Example 3.6: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then the subset $A = \{b, c\}$ is 0-sgp-closed set but it is not pre-0-closed set in $X$.

Theorem 3.7: Every 0-sgp-closed set is sgp-closed set but not converse.

Proof: It is true that $pCl(A) \subset pCl_0(A)$ for every subset $A$ of $X$.

Example 3.8: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Set $A = \{b\}$ and $U = \{a, b\}$. But $pCl(A) = X$ which is not a subset of $U$, where $U$ is semi-open in $X$. Hence $A = \{b\}$ is not 0-sgp-closed set. But it is sgp-closed set.

Theorem 3.9: Every 0-sgp-closed set is gp-closed set.

Proof: Let $A$ be an 0-sgp-closed set in a topological space $X$. Let $U$ be an open set and so it is semi-open such that $A \subseteq U$. Then $pCl(A) \subseteq U$. Hence $A$ is gp-closed set.

The converse of the above theorem need not be true as seen from the following example.

Example 3.10: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then a subset $A = \{a, b\}$ is gp-closed set but it is not 0-sgp-closed set.

Theorem 3.11: Every 0-sgp-closed set is gsp-closed set.

Proof: Let $A$ be a 0-sgp-closed set in $X$. Let $A \subseteq U$, where $U$ is open and so it is semi-open set in $X$. Then $pCl(A) \subseteq U$. But $spCl(A) \not\subseteq pCl_0(A)$. Therefore $spCl(A) \not\subseteq U$. Hence $A$ is gsp-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

Example 3.12: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then a subset $A = \{a, c\}$ is gsp-closed set and it is not 0-sgp-closed set.

Remark 3.13: The concept of 0-sgp-closed sets and 0-gs-closed sets are independent of each other as seen from the following examples.

Example 3.14: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then the subset $A = \{a, b\}$ is 0-gs-closed set but it is not 0-sgp-closed set in $X$.

Example 3.15: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{b, c\}, \{a, c\}\}$. Then the subset $A = \{c\}$ is 0-sgp-closed set but it is not 0-gs-closed set in $X$.

Remark 3.16: The notion of 0-sgp-closed sets and $\alpha$-closed sets are independent of each other as seen from the following examples.
**Example 3.17:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}\}$. Then the subset $A = \{a\}$ is $\theta$-sgp-closed set but it is not $\alpha$-closed set in $X$.

**Example 3.18:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Then the subset $A = \{b, c\}$ is $\alpha$-closed set but it is not $\theta$-sgp-closed set in $X$.

**Remark 3.19:** The concept of $\theta$-sgp-closed sets and $\alpha$gs-closed sets are independent of each other as seen from the following examples.

**Example 3.20:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{b, c\}, \{a, c\}\}$. Then a subset $A = \{c\}$ is a $\theta$-sgp-closed set but it is not $\alpha$gs-closed set.

**Example 3.21:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then a subset $A = \{a, b\}$ is $\alpha$gs-closed set but it is not $\theta$-sgp-closed set.

**Theorem 3.22:** Every $\theta$-g-closed set is $\theta$-sgp-closed set.

**Proof:** Let $A$ be a $\theta$-g-closed set in $X$. Let $A \subseteq U$, where $U$ is open set in $X$. Then $Cl(\theta)(A) \subseteq U$. But $pCl(\theta)(A) \subseteq Cl(\theta)(A)$. Therefore $pCl(\theta)(A) \subseteq \Theta U$. Hence $A$ is $\theta$-sgp-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.23:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{b, c\}, \{a, c\}\}$. Then a subset $A = \{c\}$ is $\theta$-sgp-closed set but it is not $\theta$-g-closed set.

**Remark 3.24:** The notion of $\theta$-sgp-closed sets and $\theta$-sg-closed sets are independent of each other as seen from the following examples.

**Example 3.25:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then a subset $A = \{a, b\}$ is $\theta$-sg-closed set but it is not $\theta$-sgp-closed set.

**Example 3.26:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{b, c\}, \{a, c\}\}$. Then a subset $A = \{c\}$ is $\theta$-sg-closed set but it is not $\theta$-sgp-closed set.

**Theorem 3.27:** Every $\theta$-sgp-closed set is $\theta$-gp-closed set.

**Proof:** Let $A$ be a $\theta$-sgp-closed set in $X$. Let $A \subseteq U$, where $U$ is regular-open and so it is semi-open set in $X$. Then $pCl(\theta)(A) \subseteq U$. Hence $A$ is $\theta$-gp-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.28:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then a subset $A = \{a, b\}$ is a $\theta$-gp-closed set but it is not $\theta$-sgp-closed set.

**Remark 3.29:** Union of $\theta$-sgp-closed sets need not be a $\theta$-sgp-closed set as seen from the following example.

**Example 3.30:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then the subsets $\{a\}$ and $\{b\}$ are $\theta$-sgp-closed sets but their union $\{a\} \cup \{b\} = \{a, b\}$ is not a $\theta$-sgp-closed set in $X$.

**Remark 3.31:** Intersection of $\theta$-sgp-closed sets need not be a $\theta$-sgp-closed set as seen from the following example.

**Example 3.32:** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then the subsets $\{a, b\}$ and $\{a, c\}$ are $\theta$-sgp-closed sets but their intersection $\{a, b\} \cap \{a, c\} = \{a\}$ is not a $\theta$-sgp-closed set in $X$.

**Theorem 3.33:** A set $A \subset X$ is $\theta$-sgp-open set if and only if $F \subset pInt(\theta)(A)$ whenever $F$ is semi-closed set in $X$ and $F \subset A$.

**Proof:** Necessity. Let $A$ be $\theta$-sgp-open set and $F \subset A$, where $F$ is semi-closed set. It is obvious that $A^c$ (complement of $A$) is contained in $F^c$. This implies that $pCl(\theta)(A^c) \subset F^c$. Hence $pCl(\theta)(A^c) = (pInt(\theta)(A))^c \subset F^c$, i.e. $F \subset pInt(\theta)(A)$.

Sufficiency. If $F$ is a semi-closed set with $F \subset pInt(\theta)(A)$ whenever $F \subset A$, then it follows that $A^c \subset F^c$ and $(pInt(\theta)(A))^c \subset F^c$ i.e. $pCl(\theta)(A^c) \subset F^c$. Therefore $A^c$ is $\theta$-sgp-closed set and therefore $A$ is $\theta$-sgp-open set.

**Lemma 3.34:** Let $A$ be a $\theta$-sgp-closed subset of $X$. Then,
Theorem 3.40: Let \( X \) be a topological space, \( pKer(\{x\}) \neq pKer(\{y\}) \) if and only if there exists a nonempty open set \( U \) containing \( x \) and not \( y \).

Proof: Suppose that \( X \) is pre-\( R_0 \) space and \( x, y \in X \) such that \( pKer(\{x\}) \neq pKer(\{y\}) \). Then, there exist \( z \in pCl_d(\{x\}) \) such that \( z \notin pCl_d(\{y\}) \) (or \( z \in pCl_d(\{y\}) \) such that \( z \notin pCl_d(\{x\}) \)). There exists \( V \in SO(X, \tau) \) such that \( y \notin V \) and \( z \in V \); hence \( x \in V \). Therefore, we have \( x \notin pCl_d(\{y\}) \). Thus \( x \in X \setminus pCl_d(\{y\}) \), which implies \( pCl_d(\{x\}) \subset X \setminus pCl_d(\{y\}) \) and \( pCl_d(\{x\}) \cap pCl_d(\{y\}) = \emptyset \). The proof for otherwise is similar.

Sufficiency. Let \( V \) be pre-\( R_0 \)-open set and let \( x \in V \). We will show that \( pCl_d(\{x\}) \subset V \). Let \( y \notin V \), i.e., \( y \in X \setminus V \). Then \( x \neq y \) and \( x \notin pCl_d(\{y\}) \). This shows that \( pCl_d(\{x\}) \neq pCl_d(\{y\}) \). By assumption, \( pCl_d(\{x\}) \cap pCl_d(\{y\}) = \emptyset \). Hence \( x \notin pCl_d(\{x\}) \). Therefore \( pCl_d(\{x\}) \subset V \).

Theorem 3.40: A topological space \( X \) is a pre-\( R_0 \) space if and only if for any points \( x \) and \( y \) in \( X \), \( pKer(\{x\}) \neq pKer(\{y\}) \) implies \( pKer(\{x\}) \cap pKer(\{y\}) = \emptyset \).

Proof: Suppose that \( X \) is pre-\( R_0 \) space. Thus by Lemma 3.38, for any points \( x \) and \( y \) in \( X \) if \( pKer(\{x\}) \neq pKer(\{y\}) \) then \( pCl_d(\{x\}) \neq pCl_d(\{y\}) \). Now we prove that \( pKer(\{x\}) \cap pKer(\{y\}) = \emptyset \). Assume that \( z \in pKer(\{x\}) \cap pKer(\{y\}) \). By \( z \in pKer(\{x\}) \) and Lemma 3.37, it follows that \( x \in pCl_d(\{z\}) \). Since \( x \in pCl_d(\{x\}) \), by Theorem 3.39, \( pCl_d(\{x\}) = pCl_d(\{z\}) \).
Similarly, we have \( pCl_\theta(y) = pCl_\theta(z) = pCl_\theta(x) \). This is a contradiction. Therefore, we have \( pKer_\theta(x) \cap pKer_\theta(y) = \emptyset \).

Conversely, let \( X \) be a topological space such that for any points \( x \) and \( y \) in \( X \), \( pKer_\theta(x) \neq pKer_\theta(y) \) implies \( pKer_\theta(x) \cap pKer_\theta(y) = \emptyset \). If \( pCl_\theta(x) \neq pCl_\theta(y) \), then by Lemma 3.38, \( pKer_\theta(x) \neq pKer_\theta(y) \). Because \( z \in pCl_\theta(x) \) implies that \( x \in pKer_\theta(z) \) and therefore \( pKer_\theta(x) \cap pKer_\theta(z) \neq \emptyset \). By hypothesis, we therefore have \( pKer_\theta(x) = pKer_\theta(z) \). Then \( z \in pCl_\theta(x) \cap pCl_\theta(y) \) implies that \( pCl_\theta(x) = pCl_\theta(z) = pCl_\theta(y) \). This is a contradiction. Hence, \( pCl_\theta(x) \cap pCl_\theta(y) = \emptyset \) and by Theorem 3.39, \( X \) is a pre-\( \theta \)-R_0 space.

3.41 Remark: The “Implication Diagram” about \( \theta \)-sgp-closed set.

\[
\begin{array}{cccc}
\text{Closed set} & \alpha \text{-closed set} & \text{ags-closed set} & \text{\( \theta \)-sg-closed set} & \text{gsp-closed set} \\
\text{\( \theta \)-gs-closed set} & \text{\( \theta \)-sg-closed set} & \text{sgp-closed set} & \\
\text{\( \theta \)-g-closed set} & \text{pre-\( \theta \)-closed set} & \text{gpr-closed set} & \text{gp-closed set} \\
\end{array}
\]

where \( A \rightarrow \rightarrow \rightarrow B \) (resp. \( A \longleftrightarrow \longleftrightarrow B \)) represents \( A \) implies \( B \) but not conversely (resp. \( A \) and \( B \) are independent).

4. Conclusion

In the class of \( \theta \)-sgp-closed sets defined using semi-open sets lies between the class of \( \theta \)-g-closed sets and the class of sgp-closed set. The \( \theta \)-sgp-closed set can be used to derive a new decomposition of continuity and new separation axioms. This concept can be extended to bitopological and fuzzy topological spaces.

**References**

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On θ-Semigeneralized Pre Closed Sets in Topological Spaces