Another Characterization’s of 2-Pre-Hilbert Space

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Abstract: The problem of determination necessary and sufficient conditions for a 2-normed space to be 2-pre-Hilbert space is in the focus of interest of many mathematicians. Some characterizations of 2-inner product are stated in [1], [4], [6], [7] and [12]. In this paper we gave a necessary and sufficient condition for the existence of 2-inner product in a 2-normed space \((L, \cdot, \cdot)\) applying Mercer inequality and also the generalizations of Tanaka, Kirk-Smiley and Gurarii-Sozonov results are given.

Keywords: 2-norm, 2-inner product, Dunkl-Williams inequality

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1. INTRODUCTION

Concepts of 2-norm and 2-inner product are two-dimensional analogies of the concepts of norm and inner product, respectively. The studying of 2-normed spaces and their application is simpler if 2-norm is generated by 2-inner product. On the other hand, verifying whether 2-norm is generated by a 2-inner product is not always simple. Therefore, of particular importance is the finding of different equivalent conditions of existence of 2-inner product that generates 2-norm, which is of interest in this paper.

Let \(L\) be a real vector space with dimension greater than 1 and \(\langle \cdot, \cdot \rangle\) be a real function of \(L \times L\) such that following holds true:

a) \(\langle x, y \rangle \geq 0\), for each \(x, y \in L\) and \(\langle x, y \rangle = 0\) if and only if the set \{\(x, y\)\} is linearly dependent;

b) \(\| x \cdot y \| = \| y \cdot x \|\), for each \(x, y \in L\);

c) \(\| \alpha x \cdot y \| = \| \alpha \cdot x \cdot y \|\), for each \(x, y \in L\) and for each \(\alpha \in \mathbb{R}\);

d) \(\| x + y \cdot z \| = \| x \cdot z \| + \| y \cdot z \|\), for each \(x, y, z \in L\).

The function \(\langle \cdot, \cdot \rangle\) is said to be 2-norm on \(L\), and \((L, \langle \cdot, \cdot \rangle)\) is said to be vector 2-norm space ([8]).

The above inequality \(d\) is said to be parallelepiped inequality.

Let \(n > 1\) be a positive integer, \(L\) be a real vector space, \(\dim L \geq n\) and \((\cdot, \cdot)\) be a real function of \(L \times L \times L\) such that

i) \((x, x \cdot y) \geq 0\), for each \(x, y \in L\) and \((x, x \cdot y) = 0\) if and only if \(x\) and \(y\) are linearly dependent;

ii) \((x, y \cdot z) = (y, x \cdot z)\), for each \(x, y, z \in L\);

iii) \((x, x \cdot y) = (y, x \cdot x)\), for each \(x, y \in L\);

iv) \((\alpha x, y \cdot z) = \alpha (x, y \cdot z)\), for each \(x, y, z \in L\) and for each \(\alpha \in \mathbb{R}\);

v) \((x + x_1, y \cdot z) = (x, y \cdot z) + (x_1, y \cdot z)\), for each \(x_1, x, y, z \in L\).
Function \((\cdot,\cdot)\) is said to be 2-inner product, and \((L,\langle\cdot,\cdot\rangle)\) is said to be 2-pre-Hilbert space ([4]).

R. Ehret proved that ([7]), if \((L,\langle\cdot,\cdot\rangle)\) is a 2-pre-Hilbert space, then

\[
\|x, y\| = (x, x | y)^{1/2},
\]

for each \(x, y \in L\) defines a 2-norm, so, we get vector 2-normed space \((L,\|\cdot\|)\) and thus for each \(x, y, z \in L\) the following equalities hold true

\[
\langle x, y | z \rangle = \frac{\|x+y,z\|^2 - \|x-y,z\|^2}{4},
\]

\[
\|x+y,z\|^2 + \|x-y,z\|^2 = 2(\|x,z\|^2 + \|y,z\|^2).
\]

The equality (3) in fact is an analogy of parallelogram equality and is said to be parallelepiped equality. Further, 2-normed space \(L\) is 2-pre-Hilbert if and only if for each \(x, y, z \in L\) the equality (3) holds true. The following three Lemmas we will present a three elementary statements according to the equalities (3) and (4).

**Lemma 1.** Let \((L,\|\cdot\|), \dim L > 2\) be 2-normed space. If there exists a 2-inner product \((\cdot,\cdot)_Y\), for each three-dimensional subspace \(Y\) of \(L\), such that \((y, y | z)_Y = \|y, z\|^2\), for each \(y, z \in Y\), then exists a 2-inner product \((\cdot,\cdot)_L\) such that \((x, x | z)_L = \|x, z\|^2\), for each \(x, z \in L\).

**Proof.** Let \(x, y, z \in L\). Then it exists a subspace \(Y\) of \(L\) such that \(\dim Y = 3\) and \(x, y, z \in Y\). The assumption implies that it exists 2-inner product \((\cdot,\cdot)_Y\) such that \((a, a | b)_Y = \|a, b\|^2\), for each \(a, b \in Y\). But the last in fact means that the following holds true

\[
\|x+y,z\|^2 + \|x-y,z\|^2 = (x+y, x+y | z)_Y + (x-y, x-y)_Y
= 2(x, x | z)_Y + 2(y, y | z)_Y
= 2(\|x, z\|^2 + \|y, z\|^2).
\]

Finally, the arbitrariness of \(x, y, z \in L\) implies that the parallelepiped equality holds in \(L\). It means that \(L\) is 2-pre-Hilbert space, i.e. there exists 2-inner product \((\cdot,\cdot)_L\) such that \((x, x | z)_L = \|x, z\|^2\), for each \(x, z \in L\).

**Lemma 2.** Let \((\cdot,\cdot)\) be 2-inner product in vector space \(L\) and let a linear mapping \(T: L \rightarrow L\) be injection. Then,

\[
(x,y | z)_T = (Tx,T(y) | T(z)), \quad x, y, z \in L
\]

defines 2-inner product in \(L\).

**Proof.** Let a linear mapping \(T: L \rightarrow L\) be injection, and \((\cdot,\cdot)_T\) be defined by (4). Then the following holds true

\[
(x, x | z)_T = (Tx,T(x) | T(z)) \geq 0, \quad \text{for each } x, z \in L
\]

and furthermore \((x, x | z)_T = 0\) if and only if \(T(x)\) and \(T(z)\) are linearly dependent, i.e. there exists \(\alpha, \beta \in \mathbb{R}\) such that \(\alpha \neq 0\) or \(\beta \neq 0\) and \(\alpha T(x) + \beta T(z) = 0\). The last actually means that \(T(\alpha x + \beta z) = 0\), and since \(T\) is injection, and \(T(0) = 0\) we conclude that \(\alpha x + \beta z = 0\). According to that, \((x, x | z)_T = 0\) if and only if \(x\) and \(z\) is linearly dependent, i.e. the axiom \(i\) of 2-inner product definition holds true.

Hence, \(T\) is linear mapping, and therefore the properties of 2-inner product imply that for each all \(x, y, z \in L\) and for each \(\alpha \in \mathbb{R}\) the following holds true

\[
(x, y | z)_T = (Tx,T(y) | T(z)) = (T(y),T(x) | T(z)) = (y, x | z)_T.
\]
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\[(x, x \mid y)_T = (T(x), T(x) \mid T(y)) = (T(y), T(y) \mid T(x)) = (y, y \mid x)_T,\]

\[(\alpha x, y \mid z)_T = (T(\alpha x), T(y) \mid T(z)) = (\alpha T(x), T(y) \mid T(z)) = \alpha (x, y \mid z)_T,\]

\[(x + x_1, y \mid z)_T = (T(x + x_1), T(y) \mid T(z)) = (T(x) + T(x_1), T(y) \mid T(z)) = (x, y \mid z)_T + (x_1, y \mid z)_T.\]

This means that the axioms ii)-v) of 2-inner product definition are satisfied.

**Lemma 3.** Let \(L\) and \(L_1\) be 2-pre-Hilbert spaces. Then, for the linear mapping \(F : L \to L_1\) the following holds true

\[(F(x), F(y) \mid F(z))_{L_1} = (x, y \mid z)_L, \text{ for each } x, y, z \in L\]  \hspace{1cm} (5)

if and only if

\[\|F(x), F(y)\|_{L_1} = \|x, y\|_L, \text{ for each } x, y \in L,\]  \hspace{1cm} (6)

and 2-norms on \(L\) and \(L_1\) are defined by the 2-inner products.

**Proof.** Let (5) holds for a linear mapping \(F : L \to L_1\). Then for each \(x, y \in L\) is true that

\[\|F(x, F(y))\|_{L_1} = (x, y \mid y)_L = \|x, y\|_L,\]

i.e. holds (6).

Conversely, if \(F : L \to L_1\) is a linear mapping such that holds true (6), then for each \(x, y, z \in L\)

\[(F(x), F(y) \mid F(z))_{L_1} = \frac{\|F(x) + F(y), F(z)\|_{L_1}^2 - \|F(x) - F(y), F(z)\|_{L_1}^2}{4} = \frac{\|x + y, z\|_L^2 - \|x - y, z\|_L^2}{4} = (x, y \mid z)_L,\]

i.e. (5) holds true.

In [6] C. Diminnie and A. White characterized 2-pre-Hilbert space using partial derivatives of 2-functionals, i.e. proved that if \((L, (\cdot, \cdot))\) is a 2-pre-Hilbert space in which the norm is defined by (1), then for each \(x, y, z \in L\) holds true

\[(x, y \mid z) = \lim_{t \to 0} \frac{\|x + ty, z\| - \|x, z\|}{2t}.\]

Further, the following Theorem holds true.

**Theorem 1 ([4]).** Let \((L, \| \cdot \|)\) be a 2-norm space. \(L\) is 2-pre-Hilbert space if and only if for each \(z \in L \setminus \{0\}\) one of the following conditions holds true:

\(II_1\). For each \(x, y \in L\) such that \(\|x, z\| = \|y, z\|\) and for each \(m, n \in R\) holds true

\[\|nx + my, z\| = \|nx + my, z\|;\]

\(II_2\). \(\|x + y, z\| = \|x - y, z\|\), \(x, y \in L\) implies

\[\|x + y, z\|^2 = \|x, z\|^2 + \|y, z\|^2;\]

\(II_3\). There is a real number \(\alpha \neq 0, \pm 1\) such that
\( \| x, z \| \leq \| y, z \|, \ x, y \in L \) implies \( \| x + \alpha y, z \| = \| \alpha x + y, z \| \).

II_4. There is a real number \( \alpha \neq 0, \pm 1 \) such that
\( \| x + y, z \| = \| x - y, z \|, \ x, y \in L \) implies \( \| x + \alpha y, z \| = \| x - \alpha y, z \| \).

II_5. \( \| x, z \| = \| y, z \|, \ x, y \in L \) implies that for each real number \( \alpha > 0 \) holds true
\( \| \alpha x + \alpha^{-1} y, z \| \geq \| x + y, z \| \).

II_6. For each \( x_1, x_2, x_3 \in L \) such that \( \sum_{i=1}^{3} x_i = 0 \) and \( \| x_1, z \| = \| x_2, z \| \) holds true
\( \| x_1 - x_3, z \| = \| x_2 - x_3, z \| \).

II_7. For each \( x_1, x_2, x_3, x_4 \in L \) such that \( \sum_{i=1}^{4} x_i = 0 \) and \( \| x_1, z \| = \| x_2, z \| \) and \( \| x_3, z \| = \| x_4, z \| \) holds true
\( \| x_1 - x_3, z \| = \| x_2 - x_4, z \| \) and \( \| x_2 - x_3, z \| = \| x_1 - x_4, z \| \).

II_8. The value of the expression
\[ F(x_1, x_2, x_3) = \| x_1 + x_2 + x_3, z \|^2 + \| x_1 + x_2 - x_3, z \|^2 - \| x_1 - x_2 - x_3, z \|^2 - \| x_1 - x_2 + x_3, z \|^2 \]
does not depend on \( x_3 \).

II_9. For each \( x_1, \ldots, x_n \in L, \ n \geq 3 \) such that \( \sum_{i=1}^{n} x_i = 0 \) the following holds true
\[ \sum_{i,k=1}^{n} \| x_i - x_k, z \|^2 = 2n \sum_{i=1}^{n} \| x_i, z \|^2. \]

2. **Dunkl-Williams Inequality into 2-Norm Space**

In this section we will generalize the Dunkl-Williams inequality into 2-normed space. Actually, this inequality was proven in [2], but in our further consideration we will present its proof, and also we will present a proof of the generalization of Mercer inequality (16) into 2-normed space.

**Theorem 2. a)** (Dunkl-Williams inequality). Let \( L \) be 2-normed space. Then,
\[ \| \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|}, z \| \leq \frac{4 \| x - y, z \|}{\| x, z \|^2 + \| y, z \|^2}, \tag{7} \]
for each \( z \in L \setminus \{0\} \) and for each \( x, y \in L \setminus V(z) \), where \( V(z) \) be the subspace generated by the vector \( z \).

**b)** (Mercer inequality). If \( L \) is a 2-pre-Hilbert space, then
\[ \| \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|}, z \| \leq \frac{2 \| x - y, z \|}{\| x, z \|^2 + \| y, z \|^2}, \tag{8} \]
for each \( z \in L \setminus \{0\} \) and for each \( x, y \in L \setminus V(z) \), where \( V(z) \) is the subspace generated by the vector \( z \).

**Proof.** a) Let \( L \) be 2-normed space, \( z \in L \setminus \{0\} \) and \( x, y \in L \setminus V(z) \). Then,
\[ \| x, z \| \cdot \| \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|}, z \| \leq \| x, z \| \cdot \| \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|}, z \| + \| x, z \| \cdot \| \frac{y}{\| y, z \|} - \frac{y}{\| y, z \|}, z \| \]
\[ \leq 2 \| x - y, z \| + \| y, z \| - \| x, z \| \]
\[ \leq 2 \| x - y, z \|. \]

Analogously one can prove that
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\[ \| y, z \| = \| x \|, z \| \leq 2 \| x, y, z \|. \]

Finally, by adding the last two inequalities, we get the inequality (7).

b) Let \( L \) be 2-pre-Hilbert space, \( z \in L \setminus \{0\} \) and \( x, y \in L \setminus V(z) \). Then,

\[
\| x \|, y, z \| = \left( \frac{x}{\| x, y, z \|}, \frac{y}{\| y, z \|}, z \| \right) = 2 - 2\left( \frac{x}{\| x, y, z \|} + \frac{y}{\| y, z \|} \right) = \frac{1}{\| x, y, z \|} (2 \| x, z \| + \| y, z \|) - 2(x, y \| z) = \frac{1}{\| x, y, z \|} (2 \| x, z \| + \| y, z \|) - (\| x, z \| + \| y, z \|)^2)
\]

Hence, the above equality and the parallelepiped inequality imply the following

\[
\| x - y, z \| ^2 = \left( \frac{x}{\| x, y, z \|}, y, z \| \right) = \| x - y, z \| ^2 - \left( \frac{x}{\| x, y, z \|}, y, z \| \right)^2 (\| x, z \| + \| y, z \|) - (\| x, z \| + \| y, z \|)^2) = \frac{1}{\| x, y, z \|} (\| x, z \| + \| y, z \|)^2 - (\| x, z \| + \| y, z \|)^2) \geq 0.
\]

Therefore, the inequality (8) holds true.

**Theorem 3.** Let \( L \) be a 2-normed space. The following statements are equivalent:

1) For each \( z \in L \setminus \{0\} \) and for each \( x, y \in L \setminus V(z) \), where \( V(z) \) is a subspace generated by the vector \( z \), the inequality (8) holds true.

2) If \( x, y, z \in L \) is such that \( \| x, z \| = \| y, z \| = 1 \), then

\[
\| \frac{x + y}{2}, z \| \leq (1-t) \| x + ty, z \|,
\]

for each \( t \in [0,1] \).

**Proof.** 1) \( \Rightarrow \) 2). Let suppose that the statement 1) holds true. Let \( x, y, z \in L \) be such that \( \| x, z \| = \| y, z \| = 1 \). Then \( z \in L \setminus \{0\} \) and \( x, y \in L \setminus V(z) \). Clearly, for \( t = 0 \) and \( t = 1 \), the inequality (9) holds true. If \( t \in (0,1) \), then 1) implies the following

\[
\| (1-t) x + ty, z \| = (1-t) \| x - \frac{t}{1-t} y, z \|
\]

\[
\geq \frac{1}{2} (\| x, z \| + \| y, z \|) - \frac{t}{1-t} \| x, y, z \| \]

\[
= \frac{1}{2} (1 + \frac{t}{1-t}) \| x + y, z \|
\]

i.e. the inequality (9) holds true.

2) \( \Rightarrow \) 1). Let suppose that the statement 1) holds true. Let \( z \in L \setminus \{0\} \) and \( x, y \in L \setminus V(z) \). Then, for \( \frac{x}{\| x, z \|}, \frac{y}{\| y, z \|} \in L \) holds true \( \| \frac{x}{\| x, z \|}, z \| = \| \frac{y}{\| y, z \|}, z \| = 1 \) and if we let that \( t = \frac{\| y, z \|}{\| x, z \| + \| y, z \|} \), according to 2) we get that
\[ \| \frac{x}{\|x\|} - \frac{y}{\|y\|} , z \| = 2 \frac{\| x - y \|}{\| \|x\| + \|y\|} , z \| \]
\[ \leq 2 \| (1 - \frac{\|y\|}{\|x\| + \|y\|}) \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \ | y \| , z \| \]
\[ = 2 \| x - y \| \frac{\|y\|}{\|x\| + \|y\|} , z \|. \]

i.e. the inequality (8) holds true.

**Remark 1.** In [13] it was proved that for each \( x, y, z \in L \) such that the sets \( \{x, z\} \) and \( \{y, z\} \) are linearly independent the following inequality holds true
\[ \| \frac{x}{\|x\|} - \frac{y}{\|y\|} , z \| \leq \frac{\|x - y\| + \|x\| + \|y\|}{\max\{\|x\|, \|y\|\}} . \] (10)

Hence, using the fact that for each \( x, y, z \in L \) holds true
\[ \| x, y, z \| \leq \| x - y, z \| \]
we get that (10) implies the following inequality
\[ \| \frac{x}{\|x\|} - \frac{y}{\|y\|} , z \| \leq \frac{2\|x - y\|}{\max\{\|x\|, \|y\|\}} . \] (11)

Clearly, the inequality (11), which in fact is generalization of Massera and Schäffer inequality ([15]) and holds true into an arbitrary 2-normed space is stronger than Dunkl-Williams inequality (4), but is weaker than the inequality (10).

Also, using the fact that for each \( x, y, z \in L \) holds true
\[ \| x - y, z \| + \| x, z \| - \| y, z \| \leq \sqrt{2\| x - y, z \|^2 + 2\| x, z \| - \| y, z \|} \leq 2\| x - y, z \| , \]
the inequality (10) implies that the following inequality holds true
\[ \| \frac{x}{\|x\|} - \frac{y}{\|y\|} , z \| \leq \frac{\sqrt{2\|x - y, z\|^2 + 2\|x, z\| - \|y, z\|^2}}{\max\{\|x\|, \|y\|\}} . \] (12)

Clearly, the inequality (12) is stronger than (11), but weaker than (10).

### 3. Characterizations of 2-pre-Hilbert Space

In this section we will give two characterizations of 2-inner product, which in fact are generalizations of Kirk-Smilley characterization ([10]) and Gurarii-Sozonov characterization ([9]).

**Theorem 4.** Let \( L \) be a 2-normed space. If the following inequality holds true
\[ \| \frac{x}{\|x\|} - \frac{y}{\|y\|} , z \| \leq \frac{2\|x - y\|}{\|x\| + \|y\|} , \] (13)
for each \( z \in L \setminus \{0\} \) and for each \( x, y \in L \setminus V(z) \), then \( L \) be a 2-pre-Hilbert space.

**Proof.** Let \( \alpha > 0 \), \( z \in L \setminus \{0\} \) and \( x, y \in L \setminus V(z) \) be such that \( \|x, z\| = \|y, z\| \). The inequality (13), applied to the vectors \( \alpha x \) and \( -\alpha^{-1} y \) implies the following
\[ \| \alpha x + \alpha^{-1} y , z \| \geq \| \alpha x , \alpha^{-1} y , z \| \]
\[ = \frac{\| \alpha x , z \| + \| \alpha^{-1} y , z \|}{\alpha} \]
\[ = \frac{\| \alpha x , z \| + \| \alpha^{-1} y , z \|}{2} \]
\[ = \frac{\| \alpha x + \alpha^{-1} y , z \|}{\| x, z \| + \| y, z \|} \]
\[ \geq \frac{\| x + y , z \|}{2} \]
This, according to Theorem 1 means that \( L \) is a 2-pre-Hilbert space.
Corollary 1. Let \( L \) be a 2-normed space. If \( x, y, z \in L \) be such that \( \| x, z \| = \| y, z \| = 1 \) holds true and for each \( t \in [0,1] \) holds
\[
\| \frac{x+y}{2}, z \| \leq (1-t)\| x + ty, z \|,
\]
then \( L \) is a 2-pre-Hilbert space.

Proof. The proof is a direct implication of Theorem 3 and Theorem 4.

Before we go on the following characterization of 2-pre-Hilbert space, we must mention that the condition II\(_4\) of Theorem 1 is equivalent to the following conditions:

\( II_4 \). It exists a real number \( \alpha_0 > 1 \) such that \( \| \alpha_0 x + y, z \| = \| x + \alpha_0 y, z \| \), for each \( x, y \in L \) such that \( \| x, z \| = \| y, z \| = 1 \).

\( II'_4 \). It exists a real number \( t_0 \in (0, \frac{1}{2}) \) such that
\[
\| (1-t_0)x + t_0 y, z \| = \| (1-t_0)y, z \|
\]
for each \( x, y \in L \) such that \( \| x, z \| = \| y, z \| = 1 \).

Theorem 1 and the stated above imply the validity of the following Corollary.

Corollary 2. Let \( (L, \| \cdot \|) \) be a 2-normed space. \( L \) be a 2-pre-Hilbert space if and only if for each \( z \in L \setminus \{0\} \) is satisfied one of the following conditions:

1) It exists a real number \( \alpha_0 > 1 \) such that \( \| \alpha_0 x + y, z \| = \| x + \alpha_0 y, z \| \), for each \( x, y \in L \) such that \( \| x, z \| = \| y, z \| = 1 \).

2) It exists a real number \( t_0 \in (0, \frac{1}{2}) \) such that
\[
\| (1-t_0)x + t_0 y, z \| = \| (1-t_0)y, z \|
\]
for each \( x, y \in L \) such that \( \| x, z \| = \| y, z \| = 1 \).

In the following Theorem, which actually is generalization of Tanaka result ([17]), we will prove that by weakening the conditions 1) and 2) given in Corollary 2, we get a new characterization of 2-pre-Hilbert space.

Theorem 5. Let \( (L, \| \cdot \|) \) be a 2-normed space. \( L \) is a 2-pre-Hilbert space if and only if for each \( z \in L \setminus \{0\} \) is satisfied one of the following conditions:

1) For each \( x, y \in L \) such that \( \| x, z \| = \| y, z \| = 1 \) it exists a real number \( \alpha > 1 \) such that \( \| \alpha x + y, z \| = \| x + \alpha y, z \| \).

2) For each \( x, y \in L \) such that \( \| x, z \| = \| y, z \| = 1 \) it exists a real number \( t \in (0, \frac{1}{2}) \) such that \( \| (1-t)x + ty, z \| = \| tx + (1-t)y, z \| \).

Obviously, the real numbers \( \alpha > 1 \) and \( t \in (0, \frac{1}{2}) \) can depend on \( x, y \in L \), for which \( \| x, z \| = \| y, z \| = 1 \) holds true.

Proof. Corollary 2 implies that it is sufficient to prove that the condition 2) implies that \( L \) is a 2-pre-Hilbert space, which according to Corollary 1 means that it is sufficient to prove that the condition 2) implies that \( x, y, z \in L \) is such that \( \| x, z \| = \| y, z \| = 1 \), then
\[
\| \frac{x+y}{2}, z \| \leq (1-t)\| x + ty, z \|,
\]
for each \( t \in [0,1] \).

Let \( x, y, z \in L \) be such that \( \| x, z \| = \| y, z \| = 1 \). We may assume that the set \( \{x, y\} \) is linearly independent. Consider the set
The condition 2) implies that $A \neq \emptyset$. Therefore it exists $t_0 = \sup A$. We will prove that $t_0 = \frac{1}{2}$.

Since the convexity of the function $t \to \| (1-t)x + ty, z \|$, $t \in [0,1]$, we will get that

$$\| \frac{x+y}{2}, z \| \leq \| (1-t)x + ty, z \|,$$

for each $t \in [0,1]$.

Let suppose that $t_0 < \frac{1}{2}$. Then, the continuous of 2-norm implies that $t_0 \in A$. The vectors $u = (1-t_0)x + t_0y$ and $v = t_0x + (1-t_0)y$ satisfy $\| u, z \| = \| v, z \|$. Let be $x_0 = \frac{u}{\| u, z \|}$ and $y_0 = \frac{v}{\| v, z \|}$.

The assumption implies that it exists a real number $t_1 \in (0, \frac{1}{2})$ such that

$$\| (1-t_1)x_0 + t_1y_0, z \| = \| t_1x_0 + (1-t_1)y_0, z \|.$$ 

Let $t^* = (1-t_1)t_0 + t_1(1-t_0)$. Then $t_0 < t^* < \frac{1}{2}$ and also holds

$$\| (1-t^*)x + t^* y, z \| = \| t^*x + (1-t^*)y, z \|.$$ 

This means that $t^* \in A$, and that is contradictory to $t_0 = \sup A$ and $t_0 < t^* < \frac{1}{2}$. Finally, the contradictory implies that $t_0 = \frac{1}{2}$.

**Corollary 3.** Let $(L, \| \cdot, \|)$ be a 2-normed space. $L$ be a 2-pre-Hilbert space if and only if for each $z \in L \setminus \{0\}$ is satisfied that

$$\| x + \frac{x+y}{\| x+y, z \|} z \| \leq \| y + \frac{x+y}{\| x+y, z \|} z \|, \quad (14)$$

for each $x, y \in L$ such that $\| x, z \| = \| y, z \| = 1$ and $x + y \notin V(z)$.

**Proof.** We will prove that the condition is sufficient. If $z \in L \setminus \{0\}$ and $x, y \in L$ be such that $\| x, z \| = \| y, z \| = 1$ and $x + y \notin V(z)$, then

$$\| (1+ \| x+y, z \|)x + y, z \| = (1+ \| x+y, z \|)^2 \| x, z \|^2 + 2(1+ \| x+y, z \|) \| (x, y \| z) + \| y, z \|^2$$

$$= (1+ \| x+y, z \|)^2 \| y, z \|^2 + 2(1+ \| x+y, z \|) \| (x, y \| z) + \| x, z \|^2$$

$$= \| x + (1+ \| x+y, z \|) y, z \|^2,$$

i.e. the following equality holds true

$$\| (1+ \| x+y, z \|)x + y, z \| \leq \| x + (1+ \| x+y, z \|) y, z \|, \quad (15)$$

which is equivalent to the equality (14).

We will prove that the condition is necessary. Let $z \in L \setminus \{0\}$ and $x, y \in L$ be such that $\| x, z \| = \| y, z \| = 1$ and $x + y \notin V(z)$. Then holds true the equality (14), which is equivalent to (15). But, $x + y \notin V(z)$, and therefore $1+ \| x+y, z \| > 1$. The last, according to Theorem 5, means that $L$ is a 2-pre-Hilbert space.

**Remark 2.** Since $1+ \| x+y, z \|$ depends on $x, y \in L$ such that $\| x, z \| = \| y, z \| = 1$, we can deduce that the statement given in Corollary 2 is not an implication of the statements given in Corollary 1. The last actually shows the advantage of Theorem 5.

**Example 1.** In [11] it is proved that in the set of bounded arrays of real numbers $l^\infty$ by

$$\| x, y \| = \sup_{i \neq j \in N} \left\| \frac{x_i}{y_i} x_j \right\|, \quad x = (x_i)_{i=1}^\infty, \quad y = (y_i)_{i=1}^\infty \in l^\infty.
Another Characterization’s of 2-Pre-Hilbert Space

is defined a 2-norm. The last means that \((l^p, \| \cdot \|)\) is real 2-normed space. It is easy to find that the vectors

\[ x = (1 - \frac{1}{2}, 1 - \frac{1}{2^2}, \ldots, 1 - \frac{1}{2^n}, \ldots), \quad y = (0, \frac{1}{2}, 1, \frac{1}{2^2}, 1, \ldots, \frac{1}{2^n}, 1, \ldots) \text{ and } z = (1, 0, 0, \ldots, 0, \ldots) \]

satisfy followings \(\| x, z \| = \| y, z \| = 1\) and \(x + y \notin V(z)\). Further, \(\| x + y, z \| = \frac{1}{2^2}\), and therefore

\[ x + \frac{x+y}{\| x+y, z \|} = (1 + \frac{3}{2}, 1 + \frac{3}{2^2}, \ldots, 1 + \frac{3}{2^n}, \ldots), \quad y + \frac{x+y}{\| x+y, z \|} = (2, 1, \frac{3}{2}, \ldots, \frac{3}{2^n}, 1, \ldots) \]

So,

\[ \| x + \frac{x+y}{\| x+y, z \|}, z \| = \frac{1}{4} 1 = \| y + \frac{x+y}{\| x+y, z \|}, z \|. \]

The last according to corollary 3, means that the 2-normed space \((l^p, \| \cdot \|)\) is not 2-pre-Hilbert space.

4. CONCLUSION

In example 1, by using Corollary 3 is proven that \((l^p, \| \cdot \|)\) is not 2-pre-Hilbert space. Analogously, other results obtained in this paper may find application in checking whether a 2-normed space is 2-pre-Hilbert, as is the case with spaces \((l^p(\mu, \| \cdot \|)), p > 1\).

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