International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 2, February 2015, PP 16-21
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Prime Ideals Space Theory of BL-algebras

Biao Long Meng

Department of Mathematics, Northwest University, Xi'an, P.R.China;
College of Science, Xi'an University of Science and Technology, Xi'an, P.R.China
mengbl_100@139.com

Xiao Long Xin

Department of Mathematics, Northwest University, Xi'an, P.R.China
xlxin@nwu.edu.cn

Abstract: *In this paper we study the prime ideals space of a BL-algebra, proved that the prime ideals space being a T_0 -space and also a stone space. Furthermore, we study the properties of the prime ideals space.*

Keywords: *BL-algebra, prime ideal, prime ideal space, topological space, Stone space.*

1. Introduction

The notion of BL-algebra was initiated by Hάjek [5] in order to provide an algebraic proof of the completeness theorem of Basic Logic. A well known example of BL-algebras is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. MV-algebra, Gödel-algebras and Product algebras are the most known class of BL-algebras. Cignoli et al.[3] proved that Hάjek's logic really is the logic of continuous t-norms as conjectured by Hάjek. At the same time started a systematic study of BL-algebras, and in particular, filter theory ([6],[7],[9],[11],[12]). Filter theory play an important role in studying BL-algebras. From logic point of view, various filters correspond to various sets of provable formulas. Hάjek introduced the notion of filters and prime filters in BL-algebras and proved the completeness of Basic Logic using prime filters. Another important notion of BL-algebras is ideal, which was introduced by Zhang [13]. Ideals of BL-algebras has more complex than filters, so far little literatures. But it is a very important tool to study logical algebras, Meng and Xin [8] systematically investigated the ideal theory of BL-algebras. Following a standard method [1], In the paper we study the prime ideals space and its important properties.

2. Preliminaries

Let us recall some definitions and results on BL-algebras.

Definition 2.1. An algebra $(A,\vee,\wedge,*,\to,0,1)$ of type (2, 2, 2, 2, 0, 0) is called a BL-algebra if it satisfies the following conditions:

(BL1) $(A,\vee,\wedge,0,1)$ is a bounded lattice,

(BL2) $(A,*,1)$ is a commutative monoid,

(BL3) $x*y\le z$ if and only if $x\le y\to z$ (residuation),

(BL4) $x\wedge y = x*(x\to y)$, thus $x*(x\to y) = y*(y\to x)$) (divisibility),

(BL5) $(x\to y)\vee(y\to x)=1$ (prelinearity).

In what follows, A will denote a BL-algebras, unless otherwise specified.

Definition 2.2. A nonempty subset I of a BL-algebra A is said to be an ideal of A if it satisfies:

$(I1)$ $0\in I$,

$(I2)$ $(x^- \to y^-)^- \in I$ and $x \in I$ implies $y \in I$ for all $x, y \in A$.

Obviously, $\{0\}$ and A are ideals of A. An ideal I is said to be proper if $I \neq A$.

Proposition 2.3. Let I be an ideal of a BL-algebra A. If $x \le y$ and $y \in I$ then $x \in I$.

Theorem 2.4. Let X be a nonempty subset of a BL-algebra A. Then for all $x \in A, x \in (X]$ if and only if there are $a_1, \cdots, a_n \in X$ such that $a_1^- * \cdots * a_n^- \to x^- = 1$.

Definition 2.5. Let A be a BL-algebra and I a proper ideal of A. Then I is said to be a prime ideal if $a \wedge b \in I$ implies $a \in I$ or $b \in I$ for any $a, b \in A$.

Theorem 2.6. Let A be a BL-algebra and I a proper ideal of A. Suppose S is a nonempty subset of A and $I \cap S = \varnothing$. If S is $\wedge$-closed, then there is a prime ideal M of A satisfies $I \subseteq M$ and $M \cap S = \varnothing$.

Definition 2.7. An ideal I of a BL-algebra A is said to be irreducible if, for any ideals J, K of A, $I = J \cap K$ implies $I = J$ or $I = K$.

Proposition 2.8. Let I be an ideal of a BL-algebra A. Then I is irreducible if and only if I is prime.

3. Prime Ideal Spaces

In the sequel, $I(A)$ denotes the set of all ideals of a BL-algebra A and $PI(A)$ be the set of all prime ideals. Let X be a nonempty subset of A, we define $S(X) = \{P \in PI(A) : X \not\subset P\}$ and $T(A) = \{S(I) : I \in I(A)\}$. It can be checked that $S(X) = S((X])$. If $X = \{a\}$, we write $S(a)$ instead of $S(\{a\})$.Denotes $T_0(A) = \{S(a) : a \in A\}$.

Theorem 3.1. $T(A)$ is a topology on $PI(A)$.

Proof: It is obvious that $S(0) = \varnothing$ and $S(A) = PI(A)$.

Suppose $I_\alpha \in I(A), \alpha \in \Lambda$ where Λ is a nonempty index set. Then for some $\alpha \in \Lambda$,

$$\begin{aligned} \bigcup_{\alpha \in \Lambda} S(I_\alpha) &= \{P \in PI(A) : I_\alpha \not\subset P\} \\ &= \{P \in PI(A) : \bigcup_{\alpha \in \Lambda} I_\alpha \not\subset P\} \\ &= S((\bigcup_{\alpha \in \Lambda} I_\alpha]) \end{aligned}$$

Hence $\bigcup_{\alpha \in \Lambda} S(I_\alpha) \in T(A)$. For any $I, J \in I(A)$, by proposition 2.8 we have

$$\begin{aligned} S(I) \cap S(J) &= \{P \in PI(A) : I \not\subset P\} \cap \{P \in PI(A) : J \not\subset P\} \\ &= \{P \in PI(A) : I \cap J \not\subset P\} \end{aligned}$$

Thus $S(I) \cap S(J) \in T(A)$. Therefore $T(A)$ forms a topology on $PI(A)$.

lemma 3.2. $T_0(A)$ is a base of $T(A)$.

Proof: Since for any $I \in I(A)$, we have $S(I) = \bigcup_{a \in I} S(a)$, it follows that $T_0(A)$ is a base of $T(A)$.

Recall that a ring R of sets is a nonempty set of subsets of a set S such that if $A, B \in R$, then $A \cap B \in R$ and $A \cup B \in R$. The following shows that $T_0(A)$ is a ring of sets.

Lemma 3.3. For any $a,b \in A, S(a) \cap S(b) = S(a \wedge b)$, $S(a) \cup S(b) = S(a \vee b)$, that is, $T_0(A)$ is a ring of sets.

Proof: For any $a,b \in A$, we have

$$
\begin{aligned}
P \in S(a) \bigcap S(b) &\text{ iff } P \in S(a) \text{ and } P \in S(b) \\
&\text{ iff } a \notin P \text{ and } b \notin P \\
&\text{ iff } a \wedge b \notin P \\
&\text{ iff } P \in S(a \wedge b) \\
P \in S(a) \bigcup S(b) &\text{ iff } P \in S(a) \text{ or } P \in S(b) \\
&\text{ iff } a \notin P \text{ or } b \notin P \\
&\text{ iff } a \vee b \notin P \\
&\text{ iff } P \in S(a \vee b).
\end{aligned}
$$

Hence $S(a) \cap S(b) = S(a \wedge b)$ and $S(a) \cup S(b) = S(a \vee b)$, which shows that $T_0(A)$ is a ring of sets.

Lemma 3.4. If O is a compact open subset of topological space $(T(A), PI(A))$, then $\mathrm{O} = S(a)$ for some $a \in A$.

Proof: Since O is open and $T_0(A)$ is a base of $T(A)$, there are $B \subseteq A$ such that

$$\mathrm{O} = \bigcup \{S(b) : b \in B\},$$

Noticing O is compact we have $b_1, \cdots, b_n \in B$ satisfying

$$\mathrm{O} = S(b_1) \bigcup \cdots \bigcup S(b_n).$$

By Lemma 3.3, $\mathrm{O} = S(b_1 \vee \cdots b_n)$.

Definition 3.5. A nonempty subset F of a lattice L is said to be a lattice filter if F satisfies:

(i) For any $x, y \in L$, $x \in F$ and $x \leq y$ imply $y \in F$,

(ii) $x, y \in F$ implies $x \vee y \in F$.

It is obvious that any lattice filter of a lattice is $\vee$-closed.

For any nonempty subset H of L, $<H>$ denotes the least lattice filter containing H, called as the lattice filter generated by H. It is easy to check that

$$<H> = \{x \in L : x \geq a_1 \wedge \cdots \wedge a_n, \exists a_1, \cdots, a_n \in H\}.$$

Lemma 3.6. Suppose X, Y are two nonempty subsets of A. If

$$\bigcap \{S(x) : x \in X\} \subseteq \bigcup \{S(y) : y \in Y\},$$

then there are nonempty finite subsets $X_0 \subseteq X$, $Y_0 \subseteq Y$ such that

$$\bigcap \{S(x) : x \in X_0\} \subseteq \bigcup \{S(y) : y \in Y_0\}.$$

Proof: We proceed by the following steps:

Step I. We prove $< X > \bigcap (Y] = \varnothing$. If $< X > \bigcap (Y] \neq \varnothing$, then by Theorem 2.6 there is a prime ideal P of A such that $(Y] \subseteq P$ and $P \bigcap < X >= \varnothing$. Hence $P \notin S(y)$ for all $y \in Y$ and $P \in S(x)$ for all $x \in X$, and so $P \notin \bigcup \{S(y) : y \in Y\}$ but $P \in \bigcap \{S(x) : x \in X\}$, a contradiction.

Step II. Let $z \in < X > \bigcap (Y]$. By $z \in < X >$ we know that $x_1 \wedge \cdots x_n \leq z$ for some $x_1, \cdots, x_n \in X$. Thus $S(x_1 \wedge \cdots x_n) \subseteq S(z)$. By Lemma 3.3 we have

(i) $S(x_1) \bigcap \cdots \bigcap S(x_n) \subseteq S(z)$

From $z \in (Y]$ it follows that

$$y_m^- \to (\cdots \to (y_1^- \to z^-) \cdots) = 1$$

for some $y_1, \cdots, y_m \in Y$ by Theorem 2.4. We now prove that

(ii) $S(z) \subseteq S(y_1) \bigcup \cdots \bigcup S(y_m)$.

Indeed, if there is $P \in S(z) - S(y_1) \bigcap \cdots \bigcap S(y_m)$, then $z \notin P$ but

$$P \notin S(x_1) \bigcup \cdots \bigcup S(x_n)$$

Thus $y_1, \cdots, y_m \in P$ and by

$$y_m^- \to (\cdots \to (y_1^- \to z^-) \cdots) = 1$$

we have $z \in P$, a contradiction, (ii) holds.

By (i) and (ii) we obtain $S(x_1) \bigcap \cdots \bigcap S(x_n) \subseteq S(y_1) \bigcup \cdots S(y_m)$.

Lemma 3.6 has a number of interesting corollaries.

Corollary 3.7. For any $a \in A$, $S(a)$ is a compact open subset of the topological space $PI(A)$.

Proof: In Lemma 3.6 we take $X = \{a\}$ to get the desired result.

Lemma 3.8. The family of compact open subsets of $PI(A)$ is a ring of sets and is a topological base.

Proof: This is immediate from Lemma 3.3, Lemma 3.4, Lemma 3.2 and Corollary 3.7.

By Lemma 3.4 and Corollary 3.7 we have

Corollary 3.9. An open subset O of the topological space $PI(A)$ is compact iff $O = S(a)$ for some $a \in A$.

Corollary 3.10. The topological space $PI(A)$ is compact.

Proof: It is immediate from $S(1) = PI(A)$ and Corollary 3.7.

Lemma 3.11. The topological space $PI(A)$ is a T_0-space.

Proof: Let $I, J \in PI(A)$ with $I \neq J$. Suppose that $I - J \neq \varnothing$ without loss of generality. Then $J \in S(I)$ but $I \in S(I)$. This shows that $PI(A)$ is a T_0-space .

Definition 3.12. A Stone space X is a topological space X satisfying:

(i) X is a T_0-space.

(ii) If $\{X_\alpha : \alpha \in H\}$ and $\{Y_\beta : \beta \in K\}$ are nonempty families of nonempty compact open subsets of X where H, K are two index sets and

$$\bigcup\{X_\alpha : \alpha \in H\} \subseteq \bigcap\{Y_\beta : \beta \in K\},$$

then there are finite subsets $H_0 \subseteq H$ and $K_0 \subseteq K$ such that

$$\bigcup\{X_\alpha : \alpha \in H_0\} \subseteq \bigcap\{Y_\beta : \beta \in K_0\}.$$

Theorem 3.9. The topological space $PI(A)$ is a Stone space.

Proof: It follows directly from Lemma 3.6, Corollary 3.7 and Lemma 3.11.

In what follows we discuss further properties of $PI(A)$. For any subset U of $PI(A)$, the interior, exterior and closure of U are denoted by $Int(U)$, $Ext(U)$ and $\overline{U}$ respectively, and denote

$$\Delta(U) = \bigcap\{P : P \in U\}.$$

Lemma 3.14. For any subset U of $PI(A)$ and $a \in A$, we have $a \in \Delta U$ iff $S(a) \bigcap U = \varnothing$.

Proof: It follows that

$$\begin{aligned} a \in \Delta U &\Leftrightarrow a \in P \text{ for all } P \in U \\ &\Leftrightarrow P \notin S(a) \text{ for all } P \in U \\ &\Leftrightarrow S(a) \bigcap U = \varnothing \end{aligned}$$

The proof is complete.

Theorem 3.15. For any subset U of $PI(A)$ we have

(i) $S(\Delta U) = Ext(U)$,

(ii) $\overline{U} = PI(A) - Ext(U) = \{P \in PI(A) : \Delta U \subseteq P\}$,

(iii) $Int(U) = PI(A) - \overline{Ext(U)}$.

Proof: Since

$$\begin{aligned} P \in S(\Delta U) &\Leftrightarrow \Delta U \not\subset P \\ &\Leftrightarrow a \notin P \text{ for some } a \in \Delta U \\ &\Leftrightarrow P \in S(a) \text{ for some } a \in \Delta U \\ &\Leftrightarrow P \in Ext(U) \text{ (by Lemma 3.4)}. \end{aligned}$$

it follows that $S(\Delta U) = Ext(U)$, (i) holds.

Corollary 3.16. Let U be any sunset of $PI(A)$, then U is dense in $PI(A)$ iff $\Delta(U) = \{0\}$.

Proof: Indeed, U is dense in $PI(A)$ iff $Ext(U) = \varnothing$ iff $S(\Delta U) = \varnothing$ by Theorem 3.15 (i) iff $\Delta(U) = \{0\}$.

4. CONCLUSION

In this paper we investigate further important properties of ideals of a BL-algebra. We study the prime ideals space of a BL-algebra, proved that the the prime ideals space is a T_0-space, furthermore it is also a stone space. We study the further properties of the prime ideals space, especiallly the properties of the interior, exterior and closure of U where U is a subset of $PI(A)$.

REFERENCES

[1]. Atiyah M.F, Macdonald I.G. Introduction to Commutative Algebra, Addison Wesley publishing Company, Reading, Massachussets, Menlo Park, California London Don Mills, Ontario. 1969.

[2]. Balbes R, Dwinger P. Distributive Lattice, University of Missouri Press, 1974.

[3]. Cignoli R, Esteva F, Godo L, Torrens A. Basic Fuzzy Logic is the Logic of Continuous T-norm and Their Residua, Soft Comput, 12(2000):106-112.

[4]. Gratzer G. General Lattice Theory, Academic Press, INC., New York, 1978.

[5]. Hajek P. Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.

[6]. Haveshki M, Saeid A B, Eslami E. Some filters in BL-algebras, Soft Computing, 10(2006):

[7]. 657-664.

[8]. Kondo M, Dudek W A. Filter Theory in BL-algebras, Soft Comput, 12(2008): 419-423.

[9]. Meng B.L, Xin X.L. Prime ideals and Gödel ideals of BL-algebras, Journal of advances in mathematics, 9(2015):2989-3005.

[10]. Saeid A B, Motamed S. Normal Filters in BL-Algebras, World Applied Sciences Journal 7(2009) Special Issue for Applied Math):70-76.

[11]. Saeid A B, Ahadpanah A, Torkzadeh L. Smarandache BL-Algebras, Journal of Applied Logic, 8(2010): 253-261.

[12]. Turunen E. BL-algebras of Basic Fuzzy Logic , Mathware & Soft Comp., 6(1999):49-61.

[13]. Turunen E. Boolean Deductive Systems of BL-algebras, Arch Math Logic, 40(2001): 467-473.

[14]. Zhang X H. Fuzzy logic and its algebraic analysis, Science Press, Beijing, 2008.