Rough Set Approximations on A Semi Bitopological View

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Abstract: The main goal of this paper is to introduce a new bitopological approaches to rough sets. Suggested approaches depend on two topologies, generated by a general relation. The first topology is a right topology whose subbase is the family of right neighborhoods and a subbase of the other topology, left topology, is the family of left neighborhoods, with respect to that general relation. Some Pawlak’s concepts are redefined, some properties are deduced and supported with proved propositions and many counter examples. We compare among suggested approaches, by using their approximations and accuracy measures. Hence, the best of them is determined. Finally, we deduce that traditional rough set model is a special case of any suggested model in this study.

Keywords: Rough set approximations, rough concepts, topological space, bitopological space.

1. INTRODUCTION

Rough set theory, proposed by Pawlak in 1982 [1], is a very useful mathematical tool in classification of a collected data under equivalence relation. In Pawlak’s study, any rough set is replaced by two crisp sets called lower and upper approximations of it.

Recently, many scientists have developed traditional rough set model, in many ways such as [2-13]. Especially, many interesting extensions of it have been made by using topological spaces such as [14-16].

This paper aims to introduce a bitopological space using a new bitopological near open set called semi-bi-open set $(S_{Bi})$. This bitopological space consists of two topologies. In our study, we consider that, every topology of this bitopological space is a view of the interested problem. These two topologies are generated by only one general relation, hence, there is no contradiction between these two views. Consequently, semi-bitopological rough concepts are introduced and compared with their traditional. We conclude relationships among traditional and proposed semi-bitopological approaches to rough sets in a diagram. Finally, we illustrate that, Pawlak’s model is a special case of any proposed semi bitopological approach to rough sets.

2. PRELIMINARIES

In this section, some basic definitions are introduced. Pawlak’s concepts were defined in [1] as follows.

Definition 2.1 Let $X$ be the universe set and let $E$ be an equivalence relation, representing our knowledge about the elements of $X$. Then $(X,E)$ is called Pawlak approximation space. An equivalence class of $E$ determined by element $x$ is

$$[x]_E = \{ x' \in X : E(x) = E(x') \}.$$  

Definition 2.2 Let $(X,E)$ be a Pawlak approximation space. Lower, upper and boundary approximations of a subset $A \subseteq X$ are defined as

$$E(A) = \bigcup \{ [x]_E : [x]_E \subseteq A \}, \quad \bar{E}(A) = \bigcup \{ [x]_E : [x]_E \cap A \neq \phi \} \text{ and } BND_E(A) = \bar{E}(A) - E(A).$$
Definition 2.3 Let \((X,E)\) be a Pawlak approximation space. The degree of crispness of \(A \subseteq X\) is determined by the accuracy measure, defined as
\[
\alpha_E(A) = \frac{|E(A)|}{|E(A)|} - E(A) \neq \emptyset.
\]

Definition 2.4 Let \((X,E)\) be a Pawlak approximation space and let \(x \in X\), rough membership relations to a subset \(A \subseteq X\) are defined as
\[
x \in A\text{ if } x \in E(A) \quad \text{and} \quad x \in \bar{A}\text{ if } x \in \bar{E}(A).
\]

Definition 2.5 Let \((X,E)\) be a Pawlak approximation space and let \(A, B \subseteq X\), rough inclusion relations are defined as
\[
A \subseteq B\text{ if } E(A) \subseteq E(B) \quad \text{and} \quad A \supseteq B\text{ if } \bar{E}(A) \subseteq \bar{E}(B).
\]

Topological rough approximations proposed by Wiweger [17] is the first generalization of rough set approximations based on topological structures. In his work, the lower and upper approximations are replaced by the interior and closure operators, respectively.

Definition 2.6 [17] Let \((X, \tau_i)\) be a topological space and let \(A \subseteq X\). Interior and closure operators, respectively, are
\[
int_i(A) = \bigcup \{G \in \tau_i : G \subseteq A\} \quad \text{and} \quad cl_i(A) = \bigcap \{G \in \tau_i : A \subseteq G\}.
\]
A subset \(A \subseteq X\) is called open set if \(A \in \tau_i\) and the family of all these open sets is denoted by \(O_i\). The complement of any open set is called closed set and the family of all closed sets is called \(C_i\).

Remark 2.1 Let \((X, \tau_i)\) be a topological space and let \(A \subseteq X\). If \(int_i(A) = cl_i(A)\), then \(A\) is called \(i\)-exact set, otherwise, it is called \(i\)-roughset.

Definition 2.7 [18] Let \((X, \tau_i)\) be a topological space and let \(A \subseteq X\). A subset \(A\) is called
\[\text{Semi-open set, if } A \subseteq cl_i(int_i(A)).\]
The family of all \(S_i\)-open sets is denoted by \(OS_i\). The complement of any \(S_i\)-open set is called \(S_i\)-closed set and the family of all \(S_i\)-closed sets is denoted by \(CS_i\).

Definition 2.8 [19] \((X, \tau_1, \tau_2)\) is called bitopological space, where \(\tau_1\) and \(\tau_2\) are two topologies, defined on a nonempty set \(X\). In \((X, \tau_1, \tau_2)\) a subset \(A \subseteq X\) is called
\[S_{12}\text{-open set, if } A \subseteq cl_2(int_1(A)).\]
The family of all \(S_{12}\)-open sets is denoted by \(OS_{12}\). The complement of any \(S_{12}\)-open set is called \(S_{12}\)-closed set and the family of all \(S_{12}\)-closed sets is denoted by \(CS_{12}\).

3. Semi-bi-near Rough Set Approximations
In this section, we define a new semi-bi-near open set, called \(S_{bi}\)-open set, defined on a bitopological space \((X, \tau_r, \tau_i)\) which is generated by a general relation. The subbase of the first topology \(\tau_r\) (right topology) is the family of right neighborhoods \(\{R = \{y \in X : xRy\}\}\) and the subbase of the second
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topology \( \tau_i \) (left topology) is the family of left neighborhoods (\( R_x = \{ y \in X : yRx \} \)), with respect to a relation \( R \). Relationship between traditional rough set approximations and suggested semi-bi-rough set approximations is deduced.

**Definition 3.1** Let \( (X, \tau_r, \tau_i) \) be a bitopological space and let \( A \subseteq X \). \( A \) is called

\[
S_{bi} - \text{Open set if } A \subseteq \{cl_i int_r A \cup cl_r int_i A \}.
\]

**Remark 3.1** Let \( (X, \tau_r, \tau_i) \) be a bitopological space generated by a general relation, then

(a) The complement of \( S_{bi} \)-open set is called \( S_{bi} \)-closed set.

(b) The family of all \( S_{bi} \)-open sets is denoted by \( OS_{bi} \).

(c) The family of all \( S_{bi} \)-closed sets is denoted by \( CS_{bi} \).

**Proposition 3.1** Let \( (X, \tau_r, \tau_i) \) be a bitopological space generated by a general relation, then

(a) \( O_r \subseteq OS_r \subseteq OS_{bi} \).

(b) \( O_i \subseteq OS_i \subseteq OS_{bi} \).

**Proof**

(a) \( O_r = \cup \{ A \subseteq X : A = \text{int}_r A \} \subseteq \cup \{ A \subseteq X : A \subseteq \text{cl}_i \text{ int}_r A \} = OS_r \)

\[
\subseteq \cup \{ A \subseteq X : A \subseteq \text{cl}_i \text{ int}_r A \cup \text{cl}_r \text{ int}_i A \} = OS_{bi}.
\]

We can get the proof of (b) at the same way as (a).

The following example illustrates that, containments in Proposition 3.1, may be proper.

**Example 3.1** Let \( R \) be a binary general relation defined on a nonempty set \( X = \{a,b,c,d\} \) defined by \( R = \{(a,a),(a,c),(a,d),(b,b),(b,d),(c,a),(c,b),(c,d),(d,a)\} \). Hence, the subbase of \( \tau_r \) is \{\{a,c,d\},\{b,d\},\{a,b,d\},\{a\}\} and the subbase of \( \tau_i \) is \{\{a,c,d\}, \{b,c\}, \{a,b,c\}, \{a\}\}. Then, \( \tau_r = \{X, \emptyset, \{a,c,d\}, \{a,b,d\}, \{b,d\}, \{a,d\}, \{a\}\} \) and \( \tau_i = \{X, \emptyset, \{a,c,d\}, \{a,b,c\}, \{b,c\}, \{a,c\}, \{a\}\} \). Consequently, \( OS_r = \{X, \emptyset, \{a\}, \{d\}, \{a,d\}, \{b,d\}, \{a,b,d\}, \{a,c,d\}\} \), \( OS_i = \{X, \emptyset, \{a\}, \{c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,c\}, \{a\}\} \). Hence, \( \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\} \notin OS_r \) and \( \{d\}, \{a,d\}, \{b,d\}, \{a,b,d\} \notin OS_i \). But, \( \{c\}, \{a,c\}, \{b,c\}, \{a,b,d\} \), \( \{a,d\}, \{a,b,d\}, \{b,c,d\} \in OS_{bi} \).

**Definition 3.2** Let \( (X, \tau_r, \tau_i) \) be a bitopological space generated by a general relation. For all \( i \in \{r, i\} \), topological lower (resp. topological upper) approximation of any subset \( A \subseteq X \) denoted by \( \downarrow_i A \) (resp. \( \uparrow_i A \)) and defined as follows

\[
\downarrow_i A = \cup \{ G \in \tau_i : G \subseteq A \} \quad \text{and} \quad \uparrow_i A = \cap \{ H \in \tau_i : A \subseteq H \}.
\]
Definition 3.3 Let \((X, \tau_r, \tau_l)\) be a bitopological space generated by a general relation. For all \(I \in \{S_{rl}, S_{lr}, S_{bi}\}\), \(I\)-interior (resp. \(I\)-closure) of any subset \(A \subseteq X\), denoted by \(\text{int}_I(A)\) (resp. \(\text{cl}_I(A)\)), are

\[
\text{int}_I A = \bigcup \{G \in OI : G \subseteq A\} \quad \text{and} \quad \text{cl}_I A = \bigcap \{H \in CI : A \subseteq H\}.
\]

Definition 3.4 Let \((X, \tau_r, \tau_l)\) be a bitopological space generated by a general relation. For all \(I \in \{S_{rl}, S_{lr}, S_{bi}\}\), \(I\)-lower (resp. \(I\)-upper) of \(A \subseteq X\), denoted by \(\downarrow^I A\) (resp. \(\uparrow^I A\)) are

\[
\downarrow^I A = \text{int}_I A \quad \text{and} \quad \uparrow^I A = \text{cl}_I A.
\]

Proposition 3.2 Let \((X, \tau_r, \tau_l)\) be a bitopological space, generated by a general relation, and let \(A \subseteq X\), then

\[
\begin{align*}
(a) \quad & \downarrow^r A \subseteq \downarrow^s_{rl} A \subseteq \downarrow^s_{lr} A \subseteq \downarrow^s_{bi} A \subseteq \uparrow^s_{bi} A \subseteq \uparrow^s_{rl} A \subseteq \uparrow^r A, \\
(b) \quad & \downarrow^l A \subseteq \downarrow^s_{lr} A \subseteq \downarrow^s_{bi} A \subseteq \uparrow^s_{bi} A \subseteq \uparrow^s_{lr} A \subseteq \uparrow^l A.
\end{align*}
\]

Proof

\[
\begin{align*}
(a) \quad & \downarrow^r A = \bigcup \{G \in \tau_r : G \subseteq A\} \subseteq \bigcup \{G \in OS_{rl} : G \subseteq A\} = [\downarrow^s_{rl} A] \subseteq \bigcup \{G \in OS_{bi} : G \subseteq A\} = [\downarrow^s_{bi} A] \subseteq \bigcap \{H \in CS_{bi} : A \subseteq H\} = [\uparrow^s_{bi} A] \\
& \subseteq \bigcap \{H \in \tau^l_i : A \subseteq H\} = \uparrow^r A.
\end{align*}
\]

The proof of (b) is similar to the proof of (a).

The following example illustrates that, the inverse of Proposition 3.2, does not hold.

Example 3.2 According to Example 3.1, we can create Table 1, as follows

<table>
<thead>
<tr>
<th>(A \subseteq X)</th>
<th>(\downarrow^s_{rl} A)</th>
<th>(\downarrow^s_{lr} A)</th>
<th>(\downarrow^s_{bi} A)</th>
<th>(\uparrow^s_{bi} A)</th>
<th>(\uparrow^s_{lr} A)</th>
<th>(\uparrow^s_{rl} A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{c}</td>
<td>(\emptyset)</td>
<td>({c})</td>
<td>({c})</td>
<td>({c})</td>
<td>({b,c,d})</td>
<td>({c})</td>
</tr>
<tr>
<td>{d}</td>
<td>{d}</td>
<td>(\emptyset)</td>
<td>({d})</td>
<td>({d})</td>
<td>({b,c,d})</td>
<td>({b,c,d})</td>
</tr>
<tr>
<td>{a,c}</td>
<td>{a}</td>
<td>{a,c}</td>
<td>({a,c})</td>
<td>({a,c})</td>
<td>(X)</td>
<td>({a,c})</td>
</tr>
<tr>
<td>{a,d}</td>
<td>{a,d}</td>
<td>{a}</td>
<td>{a,d}</td>
<td>{a,d}</td>
<td>(X)</td>
<td>({a,c})</td>
</tr>
<tr>
<td>{b,c}</td>
<td>(\emptyset)</td>
<td>{b,c}</td>
<td>{b,c}</td>
<td>{b,c}</td>
<td>{b,c}</td>
<td>{b,c}</td>
</tr>
<tr>
<td>{b,d}</td>
<td>{b,d}</td>
<td>(\emptyset)</td>
<td>{b,d}</td>
<td>{b,d}</td>
<td>{b,c,d}</td>
<td>{b,c,d}</td>
</tr>
<tr>
<td>{c,d}</td>
<td>{d}</td>
<td>{c}</td>
<td>{c,d}</td>
<td>{b,c,d}</td>
<td>{b,c,d}</td>
<td>{b,c,d}</td>
</tr>
<tr>
<td>{a,b,c}</td>
<td>{a}</td>
<td>{a,b,c}</td>
<td>{a,b,c}</td>
<td>{a,b,c}</td>
<td>(X)</td>
<td>{a,b,c}</td>
</tr>
<tr>
<td>{a,b,d}</td>
<td>{a,b,d}</td>
<td>{a}</td>
<td>{a,b,d}</td>
<td>{a,b,d}</td>
<td>(X)</td>
<td>{a,b,c}</td>
</tr>
<tr>
<td>{b,c,d}</td>
<td>{b,d}</td>
<td>{b,c}</td>
<td>{b,c,d}</td>
<td>{b,c,d}</td>
<td>{b,c,d}</td>
<td>{b,c,d}</td>
</tr>
</tbody>
</table>

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From Table 1, $S_{bi}$-lower approximation is the greatest lower of the studied bitopological lowers and $S_{bi}$-upper approximation is the smallest upper of the studied bitopological uppers. So, $S_{bi}$-approach is the best bitopological approach in this study.

**Proposition 3.3** Let $(X, \tau_r, \tau_l)$ be a bitopological space generated by a general relation and let $A, E \subseteq X$. For all $I \in \{S_{rl}, S_{lr}, S_{bi}\}$ we can prove the following properties

(a) $\downarrow^I X = \uparrow^I X = X$ and $\downarrow^I \emptyset = \uparrow^I \emptyset = \emptyset$.
(b) $\downarrow^I A \subseteq A \subseteq \uparrow^I A$.
(c) If $A \subseteq E$, then $\downarrow^I A \subseteq \downarrow^I E$ and $\uparrow^I A \subseteq \uparrow^I E$.
(d) $\downarrow^I A \cup E \supseteq \downarrow^I A \cup \downarrow^I E$.
(e) $\uparrow^I A \cap E \subseteq \uparrow^I A \cap \uparrow^I E$.
(f) $\downarrow^I A^c = [\uparrow^I A]^c$ and $\uparrow^I A^c = [\downarrow^I A]^c$, where $A^c$ is the complement of $A$.

**Proof** By using the properties of bitopological lower and bitopological upper approximations, defined in Definition 3.4, we get the proof, directly.

The following example illustrates that, containments in Property (c), may be proper.

**Example 3.3** According to Example 3.1, if $A = \{b\}$ and $E = \{a, b\}$, then $\downarrow_{S_{rl}}^A = \downarrow_{S_{lr}}^A = \emptyset$, $\uparrow_{S_{rl}}^A = \uparrow_{S_{lr}}^A = \{b\}$, $\downarrow_{S_{bi}}^A = \downarrow_{S_{bi}}^A = \{a, b, c\}$, $\uparrow_{S_{bi}}^A = \{a, b, d\}$ and $\uparrow_{S_{bi}}^E = \{a, b\}$. Hence, $\downarrow_{S_{rl}}^A \neq \downarrow_{S_{rl}}^E$, $\downarrow_{S_{lr}}^A \neq \downarrow_{S_{lr}}^E$, $\downarrow_{S_{bi}}^A \neq \downarrow_{S_{bi}}^E$, $\uparrow_{S_{rl}}^A \neq \uparrow_{S_{rl}}^E$, $\uparrow_{S_{lr}}^A \neq \uparrow_{S_{lr}}^E$, $\uparrow_{S_{bi}}^A \neq \uparrow_{S_{bi}}^E$.

The following example illustrates that, a containment in Property (d), may be proper.

**Example 3.4** According to Example 3.1, if $A = \{b\}$ and $E = \{c, d\}$, then $\downarrow_{S_{rl}}^A = \downarrow_{S_{lr}}^A = \emptyset$, $\downarrow_{S_{bi}}^A = \{b, c, d\}$ and $\downarrow_{S_{bi}}^E = \{b, c, d\}$. Hence, $\downarrow_{S_{rl}}^A \cup E \neq \downarrow_{S_{rl}}^A \cup \downarrow_{S_{rl}}^E$, $\downarrow_{S_{lr}}^A \cup E \neq \downarrow_{S_{lr}}^A \cup \downarrow_{S_{lr}}^E$, $\uparrow_{S_{bi}}^A \cup E \neq \uparrow_{S_{bi}}^A \cup \uparrow_{S_{bi}}^E$.

The following example illustrates that, a containment in Property (e), may be proper.

**Example 3.5** According to Example 3.1, if $A = \{a, b\}$ and $E = \{a, c, d\}$, then $\uparrow_{S_{rl}}^A = \{a, b, d\}$, $\uparrow_{S_{lr}}^A = \{a, b\}$, $\uparrow_{S_{bi}}^E = \uparrow_{S_{bi}}^E = \uparrow_{S_{bi}}^X$, $\uparrow_{S_{rl}}^A \neq \{a\}$.
\[ \uparrow_{S_{lr}} \{a\} = \{a,d\} \text{ and } \uparrow_{S_{bi}} \{a\} = \{a\}. \] Hence, \( \uparrow_{S_{lr}} A \cap E \neq \uparrow_{S_{bi}} A \cap \uparrow_{S_{lr}} E \), \( \uparrow_{S_{bi}} A \cap E \).

**Proposition 3.4** Let \((X, \tau_r, \tau_l)\) be a bitopological space, generated by a general relation and let \(A, E \subseteq X\), for all \(I \in \{S_{nr}, S_{lr}, S_{bi}\}\), the following properties hold.

(a) \( \uparrow_{I} A \cup E \supseteq \uparrow_{I} A \cup \uparrow_{I} E \).

(b) \( \downarrow_{I} A \cap E \subseteq \downarrow_{I} A \cap \downarrow_{I} E \).

**Proof** By using the properties of interior and closure operators, also from Definitions 3.3 and 3.4, we get the proof, directly.

The following example illustrates that, a containment in Property (a), may be proper. Let \(I = S_{bi}\).

**Example 3.6** According to Example 3.1, if \(A = \{a,c\}\) and \(E = \{d\}\), then \(\uparrow_{S_{bi}} A = \{a,c\}\), \(\uparrow_{S_{bi}} E = \{d\}\) and \(\uparrow_{S_{bi}} A \cup E = X\). Hence, \(\uparrow_{S_{bi}} A \cup E \neq \uparrow_{S_{bi}} A \cup \uparrow_{S_{bi}} E\).

The following example illustrates that, a containment in Property (b), may be proper. Let \(I = S_{bi}\).

**Example 3.7** According to Example 3.1, if \(A = \{b,c\}\) and \(E = \{b,d\}\), then \(\downarrow_{S_{bi}} A = \{b,c\}\), \(\downarrow_{S_{bi}} E = \{b,d\}\) and \(\downarrow_{S_{bi}} A \cap E = \emptyset\). Hence, \(\downarrow_{S_{bi}} A \cap E \neq \downarrow_{S_{bi}} A \cap \downarrow_{S_{bi}} E\).

**Proposition 3.5** Let \((X, \tau_r, \tau_l)\) be a bitopological space, generated by a general relation and let \(A \subseteq X\), for all \(I \in \{S_{nr}, S_{lr}, S_{bi}\}\), the following properties hold

(a) \( \downarrow_{I} \downarrow_{I} A = \downarrow_{I} A \).

(b) \( \uparrow_{I} \uparrow_{I} A = \uparrow_{I} A \).

**Proof**

(a) Let \(Y = \downarrow_{I} A\) and let \(u \in Y\). But, \(Y = \downarrow_{I} A = \cup \{G \in OI : G \subseteq A\}\). Then, for all \(G \subseteq A\), we have \(G \subseteq Y\), hence, \(u \in \downarrow_{I} Y\), it follows that, \(Y \subseteq \downarrow_{I} Y\). On the other hand, from Proposition 3.3, we can deduce that, \(\downarrow_{I} Y \subseteq Y\). Consequently, \(\downarrow_{I} Y = Y\). Thus, \(\downarrow_{I} \downarrow_{I} A = \downarrow_{I} A\).

(b) From Proposition 3.3, we have, \(\uparrow_{I} A^c = [\downarrow_{I} A]^c\). Then, \(\uparrow_{I} A = [\downarrow_{I} A^c]^c\) and then, \(\uparrow_{I} \uparrow_{I} A = [\downarrow_{I} \downarrow_{I} A^c]^c = [\downarrow_{I} \downarrow_{I} [\downarrow_{I} A^c]^c]^c = [\downarrow_{I} \downarrow_{I} A^c]^c\), from Property (a) of Proposition 3.5, we have, \(\downarrow_{I} \downarrow_{I} A^c = \downarrow_{I} A^c\). Hence, \(\downarrow_{I} \downarrow_{I} A^c = \downarrow_{I} A^c\). Thus, \(\uparrow_{I} \uparrow_{I} A = \uparrow_{I} A\).

**Proposition 3.6** Let \((X, \tau_r, \tau_l)\) be a bitopological space, generated by a general relation and let \(A \subseteq X\), for all \(I \in \{S_{nr}, S_{lr}, S_{bi}\}\), the following properties do not hold

(a) \( \uparrow_{I} \downarrow_{I} A = \downarrow_{I} A \).
(b) $\downarrow^I_I A = \uparrow^I_I A$.

The following example proves Property (a), of Proposition 3.6, at $I = S_{rl}$.

**Example 3.8** According to Example 3.1, if $A = \{a, c, d\}$, then $\downarrow^I_{S_{rl}} A = \{a, c, d\}$ and $\uparrow^I_{S_{rl}} \downarrow^I_{S_{rl}} A = X$. Hence, $\uparrow^I_{S_{rl}} \downarrow^I_{S_{rl}} A \neq \downarrow^I_{S_{rl}} A$.

The following example proves Property (b) of Proposition 3.6, at $I = S_{lr}$.

**Example 3.9** According to Example 3.1, if $A = \{a, b\}$, then $\uparrow^I_{S_{lr}} A = \{a, b, d\}$ and $\downarrow^I_{S_{lr}} \uparrow^I_{S_{lr}} A = \{a\}$. Hence, $\downarrow^I_{S_{lr}} \uparrow^I_{S_{lr}} A \neq \uparrow^I_{S_{lr}} \downarrow^I_{S_{lr}} A$.

Lima 3.1 Let $(X, \tau_r, \tau_l)$ be a bitopological space, generated by a general relation, and let $A, E \subseteq X$. For all $I \in \{S_{rl}, S_{lr}, S_{bi}\}$ we can prove the following property

$$[cI^c_I A] = \text{int}^I_I A^c.$$  

**Proof**

$$[cI^c_I A] = X \cap \{H \subseteq A \subseteq H \cap \tau \} = \cup (X \cap \tau \cap \{H \subseteq A \subseteq H \cap \tau \})$$

$$= \cup \{G \subseteq \text{int}^I_I A^{c} \cap \{H \subseteq A \subseteq H \cap \tau \} \} = \text{int}^I_I A^c.$$  

**Proposition 3.7** Let $(X, \tau_r, \tau_l)$ be a bitopological space, generated by a general relation, and let $A, E \subseteq X$. For all $I \in \{S_{rl}, S_{lr}, S_{bi}\}$ we can prove that

$$\downarrow^I_I A \cap E \subseteq \downarrow^I_I A \cap E.$$  

**Proof**

Where $A \cap E = A \cap E^c$, then $\downarrow^I_I A \cap E = \downarrow^I_I A \cap E = \text{int}^I_I A \cap E = \text{int}^I_I \text{int}^I_I E$. By using Lema 3.1, we get $\text{int}^I_I A \cap \text{int}^I_I E = \text{int}^I_I A \cap \text{cl}^I_I E = \text{int}^I_I A \cap \text{int}^I_I E$. But, $\text{int}^I_I E \subseteq \text{cl}^I_I E$. Consequently, $\text{int}^I_I A \cap \text{cl}^I_I E \subseteq \text{int}^I_I A \cap \text{int}^I_I E = \downarrow^I_I A \cap \downarrow^I_I E$. Hence, $\downarrow^I_I A \cap E \subseteq \downarrow^I_I A \cap \downarrow^I_I E$.

The following example illustrates that, a containment in Proposition 3.7, may be proper, at $I = S_{bi}$.

**Example 3.10** According to Example 3.1, if $A = \{a, b, c\}$ and $E = \{a, c\}$, then $\downarrow^I_{S_{bi}} A = \{a, b, c\}$, $\downarrow^I_{S_{bi}} E = \{a, c\}$ and $\downarrow^I_{S_{bi}} A \cap E = \emptyset$. Hence, $\downarrow^I_{S_{bi}} A \cap E \neq \downarrow^I_{S_{bi}} A \cap \downarrow^I_{S_{bi}} E$.

**Proposition 3.8** In a bitopological space $(X, \tau_r, \tau_l)$, generated by a general relation, for any two subsets $A, E \subseteq X$, the following property may be not satisfied for all $I \in \{S_{rl}, S_{lr}, S_{bi}\}$.
\[ \uparrow_I A - E = \uparrow_I A - \uparrow_I E. \]

Proposition 3.8 is proved by the following example, at \( I = S_{bi} \).

**Example 3.11** According to Example 3.1, if \( A = \{c,d\} \) and \( E = \{c\} \), then \( \uparrow_{S_{bi}} A = \{b,c,d\} \), \( \uparrow_{S_{bi}} E = \{c\} \) and \( \uparrow_{S_{bi}} A - E = \{d\} \). Hence, \( \uparrow_{S_{bi}} A - E \neq \uparrow_{S_{bi}} A - \uparrow_{S_{bi}} E \).

**Definition 3.5** In a bitopological space \((X, \tau^r, \tau^l)\) generated by a general relation, for all \( I \in \{r,l,S^l, S^r, S_{bi}\} \) a subset \( A \subseteq X \) is called

(a) \( I \)-Totally definable (\( I \)-exact), if \( \downarrow^I A = \uparrow^I A = A \).

(b) \( I \)-Internally definable, if \( \downarrow^I A = A \) and \( \uparrow^I A \neq A \).

(c) \( I \)-Externally definable, if \( \downarrow^I A \neq A \) and \( \uparrow^I A = A \).

(d) \( I \)-Rough, if \( \downarrow^I A \neq A \) and \( \uparrow^I A \neq A \).

The following example illustrates Definition 3.5, at \( I = S_{bi} \).

**Example 3.12** From Example 3.1, we get the following results: \( \{b\}, \{a,b\} \) are \( S_{bi} \)-externally definable sets, \( \{c,d\}, \{a,c,d\} \) are \( S_{bi} \)-internally definable sets and \( \{a\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\} \) are \( S_{bi} \)-exact sets.

**Proposition 3.9** In a bitopological space \((X, \tau^r, \tau^l)\), generated by a general relation, for any subset \( A \subseteq X \), the following properties are satisfied

(a) \( A \) is \( r \)-exact \( \rightarrow \) \( A \) is \( S^l \)-exact \( \rightarrow \) \( A \) is \( S_{bi} \)-exact.

(b) \( A \) is \( l \)-exact \( \rightarrow \) \( A \) is \( S^r \)-exact \( \rightarrow \) \( A \) is \( S_{bi} \)-exact.

(c) \( A \) is \( S_{bi} \)-rough \( \rightarrow \) \( A \) is \( S^r \)-rough \( \rightarrow \) \( A \) is \( r \)-rough.

(d) \( A \) is \( S_{bi} \)-rough \( \rightarrow \) \( A \) is \( S^l \)-rough \( \rightarrow \) \( A \) is \( l \)-rough.

**Proof**

(a) Let \( A \subseteq X \) is \( r \)-exact set, then \( \downarrow^r A = \uparrow^r A = A \). From Proposition 3.2, we have, \( \downarrow^r A \subseteq \downarrow_{S^r} A \subseteq \downarrow_{S_{bi}} A \) and \( \uparrow_{S_{bi}} A \subseteq \uparrow_{S^r} A \subseteq \uparrow^r A \). Therefore, \( \downarrow_{S^r} A = \downarrow_{S_{bi}} A = \uparrow_{S_{bi}} A = \uparrow_{S^r} A = A \). Consequently, \( A \) is \( S^r \)-exact set and \( A \) is \( S_{bi} \)-exact set.

We can get the proof of (b), (c) and (d) at the same way as (a).

**4. SEMI-BI-ROUGH CONCEPTS**

In this section, Pawlak’s concepts are redefined and studied. The relationships among suggested semi-bi-rough concepts and their traditional rough concepts. A comparison among all these approaches by using their accuracy measures is given. Finally, we conclude the relationship among all studied approaches to rough sets in a diagram.
Definition 4.1 In a bitopological space \((X, \tau_1, \tau_2)\) generated by a general relation, we can determine the degree of crispness of any subset \(A \subseteq X\), by using a bitopological accuracy measure denoted by \(C(I) A\), for all \(I \in \{r, l, S_{rl}, S_{lr}, S_{bi}\}\) and it is defined as

\[
C(I) A = \frac{j}{l} A, \quad A \neq \emptyset.
\]

Proposition 4.1 Let \((X, \tau_1, \tau_2)\) be a bitopological space and let \(A \subseteq X\), then

(a) \(0 \leq C_r A \leq C_{srl} A \leq C_{sbi} A \leq 1\).

(b) \(0 \leq C_l A \leq C_{srl} A \leq C_{sbi} A \leq 1\).

Proof By using Proposition 3.2, we get the proof directly.

The following example studies a comparison among suggested semi-bitopological approaches, by using their accuracy measures.

Example 4.1 From Example 3.1, we can create Table 2, as follows

**Table 2. Comparison among studied approaches by using their accuracy measures.**

<table>
<thead>
<tr>
<th>(C(I) A)</th>
<th>({a})</th>
<th>({c})</th>
<th>({d})</th>
<th>({a, c})</th>
<th>({a, d})</th>
<th>({b, c})</th>
<th>({b, d})</th>
<th>({c, d})</th>
<th>({a, b, c})</th>
<th>({a, b, d})</th>
<th>({b, c, d})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{srl} A)</td>
<td>1/2</td>
<td>0</td>
<td>1/3</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>2/3</td>
<td>1/3</td>
<td>1/3</td>
<td>3/4</td>
<td>2/3</td>
</tr>
<tr>
<td>(C_{srl} A)</td>
<td>1/2</td>
<td>1/3</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>2/3</td>
<td>0</td>
<td>1/3</td>
<td>3/4</td>
<td>1/3</td>
<td>2/3</td>
</tr>
<tr>
<td>(C_{sbi} A)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2/3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

From Proposition 4.1, we can deduce that, the best approach to rough sets, in this study, is \(S_{bi}\)-approach. Also, from Table 2, we notice that, by using \(S_{bi}\)-set approximations, many subsets of \(X\) become crisp, although they are not \(S_{rl}\)-exact sets or \(S_{lr}\)-exact sets.

Definition 4.2 Let \((X, \tau_1, \tau_2)\) be a bitopological space generated by a general relation and let \(A \subseteq X\). For all \(I \in \{r, l, S_{rl}, S_{lr}, S_{bi}\}\) \(I\)-rough membership relations, denoted by \(\bar{\xi}_I\) and \(\bar{\eta}_I\), are defined as

\[
x_{\bar{\xi}_I} A \quad \text{if} \quad x \in \downarrow_I A \quad \text{and} \quad x_{\bar{\eta}_I} A \quad \text{if} \quad x \in \uparrow_I A.
\]

Proposition 4.2 In a bitopological space \((X, \tau_1, \tau_2)\) generated by a general relation. For any subset \(A \subseteq X\) and for all \(I \in \{r, l, S_{rl}, S_{lr}, S_{bi}\}\), we can prove that,

(a) \(x_{\bar{\xi}_r} A \Rightarrow x_{\bar{\xi}_{srl}} A \Rightarrow x_{\bar{\xi}_{sbi}} A \Rightarrow x \in A \Rightarrow x_{\bar{\xi}_S} A \Rightarrow x_{\bar{\xi}_r} A\).

(b) \(x_{\bar{\xi}_l} A \Rightarrow x_{\bar{\xi}_{srl}} A \Rightarrow x_{\bar{\xi}_{sbi}} A \Rightarrow x \in A \Rightarrow x_{\bar{\xi}_S} A \Rightarrow x_{\bar{\xi}_l} A\).

Proof From Proposition 3.2, we get the proof, directly.

The following Example illustrates that, the inverse of Property (a), does not hold.
Example 4.2 According to Example 3.1, if  \( A = \{a, d\} \),  \( B = \{a, b\} \),  \( C = \{a, c, d\} \) and  \( E = \{b, c\} \), then \( \downarrow_{S_{bi}}^E = \emptyset \), \( \downarrow_{S_{bi}}^E = \{b, c\} \), \( \uparrow_{S_{bi}}^E = \{a\} \), \( \uparrow_{S_{bi}}^E = X \), \( \uparrow_{S_{bi}}^E = \{a, b\} \) and \( \uparrow_{S_{bi}}^E = \{a, b, c\} \), \( \downarrow_{S_{bi}}^E = \emptyset \). Hence, \( c \notin_{S_{bi}} E \) but \( c \in_{S_{bi}} E \), \( d \notin_{S_{bi}} A \) but \( d \in_{S_{bi}} A \). Then, \( \downarrow_{S_{bi}} A = \{a, b, d\} \), \( \downarrow_{S_{bi}} A = \{a\} \) and \( \downarrow_{S_{bi}} A = \{a, d\} \). Hence, \( c \notin_{S_{bi}} E \) but \( c \in_{S_{bi}} E \), \( d \notin_{S_{bi}} A \) but \( d \in_{S_{bi}} A \). Although, \( b \notin_{S_{bi}} B \) but \( b \in_{S_{bi}} B \), \( b \notin_{S_{bi}} C \) but \( b \in_{S_{bi}} C \), \( c \notin_{S_{bi}} B \) but \( c \in_{S_{bi}} B \) and \( d \notin_{S_{bi}} B \) but \( d \in_{S_{bi}} B \).

Definition 4.3 Let  \( (X, \tau_r, \tau_l) \) be a bitopological space generated by a general relation and let  \( A, E \subseteq X \). For all  \( I \in \{r, l, S_{ri}, S_{lr}, S_{bi}\} \),  \( I \)-rough inclusion relations, denoted by \( \subseteq \) and \( \supseteq \), are
\[
A \subseteq E \quad \text{if} \quad \downarrow_{S_{bi}} A \subseteq \downarrow_{S_{bi}} E \quad \text{and} \quad A \supseteq E \quad \text{if} \quad \uparrow_{S_{bi}} A \supseteq \uparrow_{S_{bi}} E.
\]
The following example illustrates Definition 4.3, at  \( I = S_{bi} \).  

Example 4.3 According to Example 3.1, if  \( A = \{a, b\} \),  \( C = \{a, c\} \),  \( D = \{b, d\} \) and  \( E = \{c, d\} \), then \( \downarrow_{S_{bi}} A = \{a\} \), \( \downarrow_{S_{bi}} C = \{a, c\} \), \( \uparrow_{S_{bi}} D = \{b, d\} \) and \( \uparrow_{S_{bi}} E = \{b, c, d\} \). Hence, \( A \subseteq C \) and \( D \supseteq E \).

Although, \( A \nsubseteq C \) and \( D \nsubseteq E \).

Remark 4.1 From our study, we can conclude the relationship among suggested approaches to rough sets of  \( A \subseteq X \) in Diagram 1, as follows

\[
\begin{array}{c}
\text{Diagram 1: Relationship among studied approaches to rough sets.} \\
\end{array}
\]

From Diagram 1, we can deduce that, any suggested semi bitopological approach to rough sets is better than its traditional. In addition,  \( S_{bi} \)-approach is the best model of proposed models in this study.

5. CONCLUSION

In this paper, proposed semi bitopological approaches, depend on a general relation. If we replace this relation by an equivalence relation  \( R \), then we get  \( S_{bi} R = \{x\} \) and then  \( \tau_r = \tau_l \), generated by  \( R \). It follows that,  \( (X, \tau_r, \tau_l) \) becomes Pawlak space  \( (X, R) \). Therefor \( \downarrow_{S_{bi}} A = R(A) \) and

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\[ \uparrow A = \bar{R}(A), \text{ for all } I \in \left\{ r, I, S_{\mu^I}, S_{\lambda^I}, S_{\nu^I} \right\}. \] It means that, in this special case, all properties and concepts of any suggested approach must be returned to their traditions.

In addition, as we proved, \( S_{bi} \) approach to rough sets is the best suggested approach in this study. Hence, by using this model, any vague concept has a big chance to be a precise concept.

ACKNOWLEDGEMENTS

We would like to acknowledge the financial support received from Shaqra University. With our sincere thanks and appreciation to Professor A.S. Salama for his support and his comments.

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