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The Similar Construction Method of Non-Homogeneous Boundary Value Problems for Second-Order Homogeneous Linear Differential Equations

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Abstract: *For a class of the non-homogeneous BVP of the second-order homogeneous linear differential equation, based on the theory of the solutions' similar structure, the impacts of each parts of the BVP (the equation and boundary value conditions) on the structure of its solutions are analyzed. A new method to solve such problems is presented which is defined as similar constructive method and the detailed steps and examples are given.*

Keywords: *Boundary value problem of differential equation, Boundary value condition, Similar kernel function, Similar structure, Similar Construction Method*

1. Introduction

In the process of solving classical seepage equations, systems of seepage equation, some kinds of second-order differential equations and the systems of second-order differential equation, their solutions of BVP have the similar structure [1-25]. However, the BVP they study is homogeneous right boundary value condition and the non-homogeneous boundary value condition. Therefore, this paper focuses on the BVP of second-order homogeneous linear differential equations which non-homogeneous the left and right boundary value conditions.

Our plan of this paper is as follows. In Sec.2 studies the similar structure of the solutions of second- order homogeneous linear differential equations are studied and the proofs for the fundamental theorem are given. In Sec.3 a new method to solve such BVP-Similar Construction Method is presented and the detailed constructed steps are derived by analyzing and utilizing the structure of the solutions. In Sec.4 an example is introduced to demonstrate the constructed process of solutions.

2. The Similar Structure of the Solutions of Boundary Value Problems

BVP in this paper is as follow:

$$\begin{cases} y''+p(x)y'+q(x)y=0 \\ [ay+(1+ab)y']_{x=\alpha}=c \\ [dy+(1+de)y']_{x=\beta}=f \end{cases} \tag{1}$$

where, a,b,c,d,e,f are both real numbers, $c^2+f^2\neq 0$, $p(x)\in C^1[\alpha,\beta], q(x)\in C^2[\alpha,\beta]$.

For inconvenient study, the previous BVP(1) is split into the following two BVP:

$$\begin{cases} y''+p(x)y'+q(x)y=0 \\ [ay+(1+ab)y']_{x=\alpha}=c \\ [dy+(1+de)y']_{x=\beta}=0 \end{cases} \tag{2}$$

and

$$\begin{cases} y''+p(x)y'+q(x)y=0 \\ [ay+(1+ab)y']_{x=\alpha}=0 \\ [dy+(1+de)y']_{x=\beta}=f \end{cases} \tag{3}$$

Where, the right value condition of the BVP (2) is homogeneous, and the light value condition of the BVP(3) is homogeneous.

Lemma 1 *(Superposition Principle) The sum solution of BVP (2) and (3) is the solution of BVP (1), vice versa.*

Proof: Suppose that $Y_1(x)$, $Y_2(x)$ are the solutions of BVP (2) and (3), let

$$Y(x)=Y_1(x)+Y_2(x) \tag{4}$$

so, take eq. (4) into the Boundary Value Problem eq. (1):

$$Y''(x)+p(x)Y'(x)+q(x)Y(x)=0 \tag{5}$$

$$aY(\alpha)+(1+ab)Y'(\alpha)=c \tag{6}$$

$$dY(\beta)+(1+de)Y'(\beta)=f \tag{7}$$

Due to eqs. (5) to (7), The sum solution of BVP(2) and (3) is the solution of BVP(1), vice versa.

From Lemma 1, solving the BVP(1) can be transformed into solving the BVP(2) and (3) respectively. The similar structure of the solutions of BVP(2) and (3) will be discussed respectively.

Lemma 2 *By two linearly independent solutions* $y_1(x)$ and $y_2(x)$ *to the equation of BVP (1), define the function* $\varphi(x,\xi)$ *:*

$$\varphi(x,\xi)=\varphi_{0,0}(x,\xi)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(\xi) & y_2(\xi) \end{vmatrix}=y_1(x)y_2(\xi)-y_2(x)y_1(\xi) \tag{8}$$

partial derivative function of $\varphi(x,\xi)$ *are :*

$$\begin{cases} \varphi_{1,0}(x,\xi)=\dfrac{\partial}{\partial x}\varphi(x,\xi)=y_1'(x)y_2(\xi)-y_2'(x)y_1(\xi) \\ \varphi_{0,1}(x,\xi)=\dfrac{\partial}{\partial \xi}\varphi(x,\xi)=y_1(x)y_2'(\xi)-y_2(x)y_1'(\xi) \\ \varphi_{1,1}(x,\xi)=\dfrac{\partial^2}{\partial x\partial \xi}\varphi(x,\xi)=y_1'(x)y_2'(\xi)-y_2'(x)y_1'(\xi) \end{cases} \tag{9}$$

Theorem 1 *If the BVP(2) has a unique solution, then the solution has the unified form as follows (i.e. similar structure of the solution):*

$$Y_1(x)=\frac{c}{a+\cfrac{1}{b+\cfrac{1}{b+\psi_1(\alpha)}}}\cdot\frac{1}{b+\psi_1(\alpha)}\cdot\psi_1(x) \tag{10}$$

where, the similar kernel function :

$$\psi_1(x)=\frac{1}{d+\cfrac{1}{e+\cfrac{\varphi_{1,0}(\alpha,\beta)}{\varphi_{1,1}(\alpha,\beta)}}}\cdot\frac{1}{e+\cfrac{\varphi_{1,0}(\alpha,\beta)}{\varphi_{1,1}(\alpha,\beta)}}\cdot[d\frac{\varphi_{0,0}(\alpha,\beta)}{\varphi_{1,1}(x,\beta)}+(1+de)\frac{\varphi_{0,1}(x,\beta)}{\varphi_{1,1}(\alpha,\beta)}] \tag{11}$$

Proof: Suppose that $y_1(x)$ and $y_2(x)$ are two linear independent solutions for the definite equation in the Boundary Value Problem eq. (2), General solution of the equation of BVP(2) :

$$Y_1(x) = C_1 y_1(x) + C_2 y_2(x) \tag{12}$$

where, C_1, C_2 are arbitrary constants which are determined by the boundary conditions of BVP(2).

By the left boundary value condition of eq. (2)

$$[ay_1(\alpha) + (1+ab)y_1'(\alpha)]C_1 + [ay_2(\alpha) + (1+ab)y_2'(\alpha)]C_2 = c \tag{13}$$

and by the right bounday valure condition

$$[dy_1(\beta) + (1+de)y_1'(\beta)]C_1 + [dy_2(\beta) + (1+de)y_2'(\beta)]C_2 = 0 \tag{14}$$

the eqs. (13) and (14) are combined to obtain a linear equations as follows:

$$\begin{pmatrix} ay_1(\alpha) + (1+ab)y_1'(\alpha) & ay_2(\alpha) + (1+ab)y_2'(\alpha) \\ dy_1(\beta) + (1+ab)y_1'(\beta) & dy_2(\beta) + (1+de)y_2'(\beta) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{15}$$

Because the Boundary Value Problem eq. (2) has a unique solution, then

$$\begin{aligned} \Delta &= [ay_1(\alpha) + (1+ab)y_1'(\alpha)][dy_2(\beta) + (1+de)y_2'(\beta)] \\ &-[ay_2(\alpha) + (1+ab)y_2'(\alpha)][dy_1(\beta) + (1+de)y_2'(\beta)] \neq 0 \end{aligned}$$

By the Lemma 2, Δ can be expressed as:

$$\begin{aligned} \Delta &= \alpha \;\; d\phi_{0,0}(\alpha,\beta) + (1+de)\phi_{0,1}(\alpha,\beta) + b[d\phi_{1,0}(\alpha,\beta) + (1+de)\phi_{1,1}(\alpha,\beta)] \\ &+ d\phi_{1,0}(\alpha,\beta) + (1+de)\phi_{1,1}(\alpha,\beta) \end{aligned} \tag{16}$$

Due to Cramer Rule, $C1$ and $C2$ are:

$$\begin{cases} C_1 = \dfrac{1}{\Delta} c[dy_2(\beta) + (1+de)y_2'(\beta)] \\ C_2 = \dfrac{-1}{\Delta} c[dy_1(\beta) + (1+de)y_1'(\beta)] \end{cases} \tag{17}$$

Substitute the eq. (17) into the eq. (12) :

$$\begin{aligned} \psi_1(x) &= \frac{d\varphi_{0,0}(x,\beta) + (1+de)\varphi_{0,1}(x,\beta)}{d\varphi_{1,0}(\alpha,\beta) + (1+de)\varphi_{1,1}(\alpha,\beta)} \\ &= \frac{1}{d + \dfrac{1}{e + \dfrac{\varphi_{1,0}(\alpha,\beta)}{\varphi_{1,1}(\alpha,\beta)}}} \cdot \frac{1}{e + \dfrac{\varphi_{1,0}(\alpha,\beta)}{\varphi_{1,1}(\alpha,\beta)}} \cdot \left[d\,\frac{\varphi_{0,0}(\alpha,\beta)}{\varphi_{1,1}(x,\beta)} + (1+de)\frac{\varphi_{0,1}(x,\beta)}{\varphi_{1,1}(\alpha,\beta)} \right] \end{aligned} \tag{18}$$

The eq. (10) can be obtained after reducing and sorting the eq. (18).

Corollary 1 *The first fraction of eq. (10) is solution of the Boundary Value Problem (2) which left boundary value condition is changed as following:*

$$\left[y(x) + by'(x)\right]_{x=\alpha} = \frac{c}{a + \dfrac{1}{b + \psi_1(\alpha)}} \tag{19}$$

Theorem 2 *If the Boundary Value Problem eq. (3) has a unique solution, then the solution has a unified form (i.e. the solution has a similar structure):*

$$Y_2(x) = \frac{f}{d + \dfrac{1}{e + \psi_2(\beta)}} \cdot \frac{1}{e + \psi_2(\beta)} \cdot \psi_2(x) \tag{20}$$

where, the similar kernel function:

$$\psi_2(x) = \frac{1}{a + \cfrac{1}{b + \cfrac{\varphi_{0,1}(\alpha,\beta)}{\varphi_{1,1}(\alpha,\beta)}}} \cdot \frac{1}{b + \cfrac{\varphi_{0,1}(\alpha,\beta)}{\varphi_{1,1}(\alpha,\beta)}} \cdot \left[a\frac{\varphi_{0,0}(\alpha,x)}{\varphi_{1,1}(\alpha,\beta)} + (1+ab)\frac{\varphi_{1,0}(\alpha,x)}{\varphi_{1,1}(\alpha,\beta)} \right] \tag{21}$$

The proof of the Theorem 2 is the similar to the Theorem 1.

Corollary 2 *The first fraction of eq. (20) is solution of the Boundary Value Problem (2) which right boundary value condition is changed as following:*

$$\left[y(x) + ey'(x)\right]_{x=\beta} = \frac{f}{d + \cfrac{1}{e + \psi_2(\beta)}} \tag{22}$$

By the lemma 1 and the theorems 1 and 2, it is not difficult to get the following theorem.

Theorem 3 *If the BVP(1) has a unique solution, then the solution has the structure as follows:*

$$Y(x) = \frac{c}{a + \cfrac{1}{b + \psi_1(\alpha)}} \cdot \frac{1}{b + \psi_1(\alpha)} \cdot \psi_1(x) + \frac{f}{d + \cfrac{1}{e + \psi_2(\beta)}} \cdot \frac{1}{e + \psi_2(\beta)} \cdot \psi_2(x) \tag{23}$$

where, $\psi_1(x)$ and $\psi_2(x)$ are defined in the eqs. (9) and (21).

3. THE SIMILAR CONSTRUCTION METHOD OF SOLVING BVP(1)

1. Through observing and analyzing the similar structure eqs. (10), (20) and (23) and the similar kernel function eqs. (11) and (21), the similar structures eqs. (10), (20) and (23) have the structural similarity and contain the similar kernel function $\psi_i(t)$ (i = 1, 2).
2. From the eqs. (11), (18) and (20),the similar kernel function $\psi_i(t)$ is obtained by the guide function $\varphi(x,\xi)$ and the left(right) boundary value condition .
3. From the eqs. (8) and (9),the guide function $\varphi(x,\xi)$ can be obtained by two linearly independent solutions of the equation of BVP, and the generating function of solutions is got by differentiating the guide function $\varphi(x,\xi)$.
4. The theorem 3 shows that the coefficients of left and right boundary value conditions are assembled to construction the solutions of the BVP(1), which no longer needs complicated calculations.

By analyzing the above items, it is easy to obtain a new method-Similar Construction Method for solving the non-homogeneous BVP (1). Without the complicated derivation and proof, we just by two linearly independent solutions of the Boundary Value Problem equation to obtain the solution of the equation of BVP (1),and the coefficients of boundary value conditions ,the solution of BVP (1) is constructed as follows steps:

Step 1. The eq. (8) is viewed as the function of guide $\varphi(x,\xi)$ by two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the Boundary Value Problem equations.

Step 2. The generating function of solutions is obtained because of the eq. (9) and the function of guide $\varphi(x,\xi)$.

Step 3. The similar kernel function $\psi_1(x)$ and $\psi_2(x)$ are generated by the generating function of solutions $\varphi_{i,j}(x,\xi)$ (i, j = 0,1) and the coefficients of boundary value conditions a, b, c and d, then $\psi_1(\alpha)$ and $\psi_2(\beta)$ are calculated.

Step 4. The coefficients a, b, c, d, e and f of boundary value conditions and $\psi_1(x)$, $\psi_1(\alpha)$,$\psi_2(x)$ and $\psi_2(\beta)$ are assembled to obtain the solution $Y(x)$ of the Boundary Value Problem eq. (1) by using the eq. (23).

4. APPLICATION EXAMPLE

In this section, we will solve the following BVP[25] with Similar Construction Method .

$$\begin{cases} \dfrac{d^2\bar{y}}{dr_D^2} - z\bar{y} = 0 & (24) \\ [(C_D z + 1 + \alpha)\bar{y} - \dfrac{d\bar{y}}{dr_D}]|_{r_D=1} = -\dfrac{\alpha}{z} & (25) \\ \lim\limits_{r_D\to\infty} \bar{y} = 0, or\ \bar{y}|_{r_D=R_D} = 0, or\ (r_D \dfrac{d\bar{y}}{dr_D} - \bar{y})|_{r_D=R_D} = 0 & (26) \end{cases}$$

Where, In this paper, the constant pressure of the outside pool boundary is just proved for the Boundary Value Problem eqs.(24), (25) and (26) and the infinity and closed outside pool boundary are the same as above above.

When outer boundary condition is constant pressure, i.e. the right boundary value condition $\bar{y}\big|_{r_D=R_D} = 0$, the solution steps can be constructed as following.

Step 1. The eq. (8) is viewed as the function of guide $\varphi(x,\xi)$ by two linearly independent solutions $e^{\sqrt{z}r_D}$ and $e^{-\sqrt{z}r_D}$ of the eq.(24).

$$\varphi(r_D,\xi) = \varphi_{0,0}(r_D,\xi) = -2\sinh\sqrt{z}(\xi - r_D) \tag{27}$$

Step 2. The generating function of one solution is obtained because of the eq. (15) and the function of guide $\varphi(x,\xi)$.

$$\varphi_{1,0}(r_D,\xi) = 2\sqrt{z}\cosh\sqrt{z}(\xi - r_D) \tag{28}$$

Step 3. The left boundary value condition eq.(25) is appropriately deformed as:

$$[-(C_D Z + 1 + \alpha)\bar{y} + \frac{d\bar{y}}{dr_D}]\Big|_{r_D=1} = \frac{\alpha}{Z} \tag{29}$$

We compare it with the Boundary Value Problem eq. (1) and let

$$a = -(C_D z + 1 + \alpha), b = 0, c = \frac{\alpha}{z}, d = 1, e = 0, f = 0,$$

then can obtain the similar kernel function:

$$\psi_1(r_D,z) = \frac{\varphi_{0,0}(r_D,R_D)}{\varphi_{1,0}(r_D,R_D)|_{r_D=1}} = \frac{-\sinh\sqrt{z}(R_D - r_D)}{\sqrt{z}\cosh\sqrt{z}(R_D - 1)} \tag{30}$$

and

$$\psi_1(1,z) = -\frac{1}{\sqrt{z}}\tanh\sqrt{z}(R_D - 1) \tag{31}$$

Step 4. The coefficients of boundary value conditions and the similar kernel function are assembled to obtain:

$$\bar{y}(r_D) = -\frac{\alpha}{z}\cdot\frac{a}{(C_D z + 1 + \alpha) - \dfrac{1}{\psi_1(1,Z)}}\cdot\frac{1}{\psi_1(1,z)}\cdot\psi_1(r_D,z) \tag{32}$$

And

$$\Phi(r_D, z) = -\psi_1(r_D, z) \tag{33}$$

So, we get the result:

$$\bar{y}(r_D) = -\frac{\alpha}{z}.\frac{a}{(C_D z + 1 + \alpha) - \dfrac{1}{\Phi_1(1,z)}}.\frac{1}{\Phi_1(1,z)}.\Phi_1(r_D, z) \tag{34}$$

The eq. (34) is the same as the equation in the reference[26]. However, the similar construction method is used for solving which avoid the complex calculations. Due to the clear-cut solving steps of this method, if this method is applied to the analysis software, it will play a multiplier effect.

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