A Proof of "Goldbach's Conjecture"

R Ellman

The-Origin Foundation, Inc. Santa Rosa, California United States of America <u>RogerEllman@The-Origin.org</u>

Abstract: *Presented is a proof of "Goldbach's Conjecture" that is simple, direct, and concise and that is readily understandable by most mathematically or scientifically trained persons.*

Keywords: Goldbach, mathematics, proof

1. INTRODUCTION

Goldbach's Conjecture states:

Every even number greater than two can be expressed as the sum of two primes.

2. Step 1

General

All of the prime numbers other than 2 are odd. The sum of any two of those odd prime numbers is always an even number. Therefore, it only remains to show that the combinations* of all prime numbers other than 2, taken two at a time, summed in pairs, yields all of the even numbers greater than 4. That, along with that the even number 4 is the sum of the pair of prime numbers [2 + 2] will complete the proof. Goldbach's Conjecture has been verified up to 10^8 by numerical calculations.¹

*[Here the combinations in pairs may include the same number twice.]

3. STEP 2

How combinations of primes summed in pairs might yield all even numbers.

The odd prime numbers comprise a string of odd numbers each greater than the prior by two except that there are various gaps [intervals of one or more non-primes] in the sequence. For example, all of the odd numbers < 200 are as in equation (1) with the non-primes in bold face [the number 1, which is divisible only by 1 and itself, is nevertheless defined as non-prime].

1	3	5	7	9	11	13	15	17	19	21	(1)
23	25	27	29	31	33	35	37	39	41	43	()
45	47	49	51	53	55	57	59	61	63	65	
67	69	71	73	75	77	79	81	83	85	87	
89	91	93	95	97	99	101	103	105	107	109	
111	113	115	117	119	121	123	125	127	129	131	
133	135	137	139	141	143	145	147	149	151	153	
155	157	159	161	163	165	167	169	171	173	175	
177	179	181	183	185	187	189	191	193	195	197	
199											

Designating the set of all prime numbers to be $\{P_i, i = 1, 2, \infty ...\} = 3, 5, 7, not 9, 11, ...,$ the first sub-set of the set of all combinations* of P_i taken and summed in pairs is the sub-set **©ARC** Page 45 $\{3 + P_{i}\}$. That sub-set produces some, but not all, of the even numbers generated as the sum of two primes as in equation (2). Bold face indicates gaps in the even numbers sequence, even numbers that are the sum of two odd numbers one or both of which is non-prime and therefore do not satisfy the conjecture.

Sub-Set {	3 + E	P _i } =								
4	6	8	10	12	14	16	18	20	22	24
26	28	30	32	34	36	38	40	42	44	46
48	50	52	54	56	58	60	62	64	66	68
70	72	74	76	78	80	82	84	86	88	90
92	94	96	98	100	102	104	106	108	110	112
114	116	118	120	122	124	126	128	130	132	134
136	138	140	142	144	146	148	150	152	154	156
158	160	162	164	166	168	170	172	174	176	178
180	182	184	186	188	190	192	194	196	198	200

The gaps in the sequence of even numbers generated by the sub-set $\{3 + P_{i}\}\)$ are due to the gaps in the sequence of primes $\{P_{i}\}\)$ per equation (1), above. The next sub-set, $\{5 + P_{i}\}\)$, fills in some of those gaps while leaving corresponding other ones, equation (3), again in **bold** face.

Sub-Set $\{5 + P_i\} =$

6	8	10	12	14	16	18	20	22	24	26
28	30	32	34	36	38	40	42	44	46	48
50	52	54	56	58	60	62	64	66	68	70
72	74	76	78	80	82	84	86	88	90	92
94	96	98	100	102	104	106	108	110	112	114
116	118	120	122	124	126	128	130	132	134	136
138	140	142	144	146	148	150	152	154	156	158
160	162	164	166	168	170	172	174	176	178	180
182	184	186	188	190	192	194	196	198	200	202
204										

Applying the two sub-sets together, however, the number of gaps in the sequence of all even numbers that are generated as the sum of two primes is reduced. Sub-set $\{5 + P_{i}\}$ generates some even numbers that are the sum of two primes that $\{3 + P_{i}\}$ does not. The combined effect is as in equation (4), for which if a number is not bold in one or both of equation 2 and 3 then it is not bold below and is not a gap.

The even	numbe	ers [≤	200]	as g	enera	ted b	У				
Sub-Sets	{3+P	' _i } an	d {5+	P _i }	combi	ned =					
4	6	8	10	12	14	16	18	20	22	24	
26	28	30	32	34	36	38	40	42	44	46	
48	50	52	54	56	58	60	62	64	66	68	
70	72	74	76	78	80	82	84	86	88	90	
92	94	96	98	100	102	104	106	108	110	112	
114	116	118	120	122	124	126	128	130	132	134	
136	138	140	142	144	146	148	150	152	154	156	
158	160	162	164	166	168	170	172	174	176	178	
180	182	184	186	188	190	192	194	196	198	200	

The sub-set $\{5 + P_i\}$ supplies a missing even number wherever it encounters the beginning of a gap in the sequence of even numbers that were generated by the $\{3 + P_i\}$ sub-set. That happens because each number in $\{5 + P_i\}$ is 2 more [one odd number higher] than the corresponding number in $\{3 + P_i\}$. The effect is that any number in equation (2) that is at the beginning of a gap [bold face with non-bold face to its left] moves one position to the left in equation (3), moves to a position not in a gap [non-bold face]. The only exception is the initial gap, 4, which has already been addressed.

Because of this, were the sequence 3, 5, ... of the subsets $\{3 + P_i\}$, $\{5 + P_i\}$, ... to continue without any breaks, for example were it to proceed $\{7 + P_i\}$, $\{9 + P_i\}$, $\{11 + P_i\}$, ... then all of those sub-sets collectively would eventually fill in all of the gaps in the original sequence generated by $\{3 + P_i\}$ and generate all of the even numbers as sums of two primes. That is, that would happen

(4)

Page 46

(3)

A Proof of "Goldbach's Conjecture"

provided that each of the gaps is such that there are more odd numbers, 5, 7, 9, 11, ... (but not 3, which generates the original gaps), preceding the gap than there are even numbers in the gap.

However, the sequence of subsets has breaks such as the $\{ \mathbf{g} + P_{\underline{i}} \}$ sub-set, which generates no even numbers as the sum of two primes because one of the two numbers summed is always the non-prime \mathbf{g} . There now remain two issues: what of the invalid sub-sets, ones that produce breaks in the sub-set sequence such as $\{ \mathbf{g} + P_{\underline{i}} \}$, and are there always sufficient prime [not merely odd] numbers preceding each gap?

4. STEP 3

<u>The invalid sub-sets such as sub-set</u> $\{ \mathbf{9} + P_i \}$.

The equations below present the even numbers generated by each of sub-sets $\{7 + P_{i}\}$ through $\{17 + P_{i}\}$ for which the index, *I*, where I = 7, *9*, 11, 13, 17, ..., is prime (except *9*) and for each presents the cumulative effect of all of the sub-sets through that point in eliminating gaps.

```
(5)
Sub-Set \{7 + P_i\} =
         8
             10
                         14
                   12
                               16
                                     18
                                           20
                                                 22
                                                       24
                                                            26
                                                                  28
        30
              32
                   34
                         36
                               38
                                     40
                                           42
                                                 44
                                                       46
                                                             48
                                                                  50
        52
              54
                   56
                         58
                               60
                                     62
                                           64
                                                 66
                                                       68
                                                            70
                                                                  72
        74
              76
                  78
                        80
                               82
                                     84
                                           86
                                                 88
                                                       90
                                                            92
                                                                  94
        96
             98 100
                        102
                             104
                                    106
                                          108
                                                110
                                                     112
                                                           114
                                                                 116
       118
            120
                  122
                        124
                              126
                                    128
                                          130
                                                132
                                                     134
                                                           136
                                                                 138
       140
            142
                  144
                        146
                              148
                                    150
                                          152
                                                154
                                                     156
                                                           158
                                                                 160
            164
                  166
                        168
                              170
                                          174
                                                           180
                                                                 182
       162
                                    172
                                                176
                                                     178
       184
            186
                 188
                        190
                              192
                                    194
                                         196
                                               198
                                                     200
                                                            ...
                                                                                     (6)
The even numbers [\leq 200] as generated by Sub-Sets
\{3 + P_i\}, \{5 + P_i\} \text{ and } \{7 + P_i\} \text{ combined} =
         4
               6
                    8
                         10
                                                                  24
                               12
                                     14
                                           16
                                                 18
                                                       20
                                                            22
        26
              28
                    30
                         32
                               34
                                     36
                                           38
                                                 40
                                                       42
                                                             44
                                                                  46
        48
                         54
                               56
                                     58
              50
                   52
                                           60
                                                 62
                                                       64
                                                             66
                                                                  68
        70
              72
                   74
                         76
                               78
                                     80
                                           82
                                                 84
                                                            88
                                                                  90
                                                       86
        92
                         98
              94
                   96
                             100
                                    102
                                          104
                                                106
                                                     108
                                                           110
                                                                 112
                                    124
                                         126
                                               128
                        120
                              122
                                                     130
                                                           132
                                                                 134
       114
            116
                  118
       136
            138
                  140
                        142
                              144
                                    146
                                          148
                                                150
                                                     152
                                                           154
                                                                 156
       158
            160
                  162
                        164
                              166
                                    168
                                          170
                                                172
                                                     174
                                                           176
                                                                 178
       180
            182
                  184
                        186
                              188
                                    190
                                          192
                                                194
                                                     196
                                                           198
                                                                 200
Sub-Set \{11 + P_{i}\} =
                                                                                     (7)
        12
              14
                   16
                         18
                               20
                                     22
                                           24
                                                       28
                                                 26
                                                             30
                                                                  32
        34
              36
                   38
                         40
                               42
                                     44
                                           46
                                                 48
                                                       50
                                                             52
                                                                  54
                   60
                         62
                               64
                                                 70
                                                       72
                                                            74
        56
              58
                                     66
                                           68
                                                                  76
        78
              80
                   82
                         84
                               86
                                     88
                                           90
                                                 92
                                                       94
                                                            96
                                                                  98
       100
            102
                  104
                        106
                             108
                                    110
                                          112
                                                114
                                                     116
                                                           118
                                                                 120
                                    132
       122
            124
                  126
                        128
                              130
                                          134
                                                136
                                                     138
                                                           140
                                                                 142
       144
            146
                  148
                        150
                              152
                                    154
                                          156
                                                158
                                                     160
                                                           162
                                                                 164
             168
                  170
                        172
                              174
                                    176
                                          178
                                                180
                                                     182
                                                           184
                                                                 186
       166
       188
            190
                  192
                        194
                              196
                                    198
                                          200
                                                 ...
The even numbers [\leq 200] as generated by Sub-Sets
                                                                                     (8)
```

 $\{3 + P_i\}, 5 + P_i\}, \{7 + P_i\}$ and $\{11 + P_i\}$ combined =

R Ellman

	4	6	8	10	12	14	16	18	20	22	24		
	26	28	30	32	34	36	38	40	42	44	46		
	48	50	52	54	56	58	60	62	64	66	68		
	70	72	74	76	78	80	82	84	86	88	90		
	92	94	96	98	100	102	104	106	108	110	112		
	114	116	118	120	122	124	126	128	130	132	134		
	136	138	140	142	144	146	148	150	152	154	156		
	158	160	162	164	166	168	170	172	174	176	178		
	180	182	184	186	188	190	192	194	196	198	200		
							-	-					
Sub-Se	et {1	3 + F	'i} =										(9)
	14	16	18	20	22	24	26	28	30	32	34		
	36	38	40	42	44	46	48	50	52	54	56		
	58	60	62	64	66	68	70	72	74	76	78		
	80	82	84	86	88	90	92	94	96	98	100		
	102	104	106	108	110	112	114	116	118	120	122		
	124	126	128	130	132	134	136	138	140	142	144		
	146	148	150	152	154	156	158	160	162	164	166		
	168	170	172	174	176	178	180	182	184	186	188		
	190	192	194	196	198	200		102	104	100	100		
	190	192	ТЭч	190	190	200	•••						
The ev	ven n	umber	s [≤	200]	as ge	nerat	ed by	Sub-	Sets				(10)
{3+Pi										mbine	d =		
-													
	4	6	8	10	12	14	16	18	20	22	24		
	26	28	30	32	34	36	38	40	42	44	46		
	48	50	52	54	56	58	60	62	64	66	68		
	70	72	74	76	78	80	82	84	86	88	90		
	92	94	96	98	100	102	104	106	108	110	112		
	114	116	118	120	122	124	126	128	130	132	134		
	136	138	118 140	120 142	122 144	124 146	126 148	128 150	130 152	132 154	134 156		
	136 158	138 160	118 140 162	120 142 164	122 144 166	124 146 168	126 148 170	128 150 172	130 152 174	132 154 176	134 156 178		
	136	138	118 140	120 142	122 144	124 146	126 148	128 150	130 152	132 154	134 156		
Sub-Se	136 158 180	138 160 182	118 140 162 184	120 142 164	122 144 166	124 146 168	126 148 170	128 150 172	130 152 174	132 154 176	134 156 178		(11)
Sub-Se	136 158 180 et {1	138 160 182 7 + F	118 140 162 184 ?i} =	120 142 164 186	122 144 166 188	124 146 168 190	126 148 170 192	128 150 172 194	130 152 174 196	132 154 176 198	134 156 178 200		(11)
Sub-Se	136 158 180 et {1 18	138 160 182 7 + F 20	118 140 162 184 P_1 = 22	120 142 164 186	122 144 166 188 26	124 146 168 190 28	126 148 170 192 30	128 150 172 194 32	130 152 174 196 34	132 154 176 198 36	134 156 178 200 38		(11)
Sub-Se	136 158 180 et {1 18 40	138 160 182 7 + F 20 42	118 140 162 184 P_i = 22 44	120 142 164 186 24 46	122 144 166 188 26 48	124 146 168 190 28 50	126 148 170 192 30 52	<pre>128 150 172 194 32 54</pre>	130 152 174 196 34 56	132 154 176 198 36 58	134 156 178 200 38 60		(11)
Sub-Se	136 158 180 et {1 18 40 62	138 160 182 7 + F 20 42 64	118 140 162 184 ?i} = 22 44 66	120 142 164 186 24 46 68	122 144 166 188 26 48 70	124 146 168 190 28 50 72	126 148 170 192 30 52 74	128 150 172 194 32 54 76	130 152 174 196 34 56 78	132 154 176 198 36 58 80	134 156 178 200 38 60 82		(11)
Sub-Se	136 158 180 et {1 18 40 62 84	138 160 182 7 + F 20 42 64 86	118 140 162 184 2: 2 44 66 88	120 142 164 186 24 46 68 90	122 144 166 188 26 48 70 92	124 146 168 190 28 50 72 94	126 148 170 192 30 52 74 96	 128 150 172 194 32 54 76 98 	130 152 174 196 34 56 78 100	132 154 176 198 36 58 80 102	134 156 178 200 38 60 82 104		(11)
Sub-Se	136 158 180 et {1 18 40 62 84 106	138 160 182 7 + F 20 42 64 86 108	118 140 162 184 22 44 66 88 110	120 142 164 186 24 46 68 90 112	122 144 166 188 26 48 70 92 114	124 146 168 190 28 50 72 94 116	126 148 170 192 30 52 74 96 118	128 150 172 194 32 54 76 98 120	130 152 174 196 34 56 78 100 122	132 154 176 198 36 58 80 102 124	134 156 178 200 38 60 82 104 126		(11)
Sub-Se	136 158 180 et {1 18 40 62 84 106 128	138 160 182 7 + F 20 42 64 86 108 130	118 140 162 184 22 44 66 88 110 132	120 142 164 186 24 46 68 90 112 134	122 144 166 188 26 48 70 92 114 136	124 146 168 190 28 50 72 94 116 138	126 148 170 192 30 52 74 96 118 140	128 150 172 194 32 54 76 98 120 142	130 152 174 196 34 56 78 100 122 144	132 154 176 198 36 58 80 102 124 146	134 156 178 200 38 60 82 104 126 148		(11)
Sub-Se	136 158 180 et {1 18 40 62 84 106 128 150	138 160 182 7 + F 20 42 64 86 108 130 152	118 140 162 184 22 44 66 88 110 132 154	120 142 164 186 24 46 68 90 112 134 156	122 144 166 188 26 48 70 92 114 136 158	124 146 168 190 28 50 72 94 116 138 160	126 148 170 192 30 52 74 96 118 140 162	128 150 172 194 32 54 76 98 120 142 164	130 152 174 196 34 56 78 100 122 144 166	132 154 176 198 36 58 80 102 124 146 168	134 156 178 200 38 60 82 104 126 148 170		(11)
Sub-Se	136 158 180 et {1 18 40 62 84 106 128 150 172	138 160 182 7 + F 20 42 64 86 108 130 152 174	118 140 162 184 22 44 66 88 110 132 154 176	120 142 164 186 24 46 68 90 112 134 156 178	122 144 166 188 26 48 70 92 114 136 158 180	124 146 168 190 28 50 72 94 116 138	126 148 170 192 30 52 74 96 118 140 162	128 150 172 194 32 54 76 98 120 142	130 152 174 196 34 56 78 100 122 144	132 154 176 198 36 58 80 102 124 146	134 156 178 200 38 60 82 104 126 148		(11)
Sub-Se	136 158 180 et {1 18 40 62 84 106 128 150	138 160 182 7 + F 20 42 64 86 108 130 152	118 140 162 184 22 44 66 88 110 132 154	120 142 164 186 24 46 68 90 112 134 156	122 144 166 188 26 48 70 92 114 136 158	124 146 168 190 28 50 72 94 116 138 160	126 148 170 192 30 52 74 96 118 140 162	128 150 172 194 32 54 76 98 120 142 164	130 152 174 196 34 56 78 100 122 144 166	132 154 176 198 36 58 80 102 124 146 168	134 156 178 200 38 60 82 104 126 148 170		(11)
	136 158 180 et {1 18 40 62 84 106 128 150 172 194	138 160 182 7 + F 20 42 64 86 108 130 152 174 196	<pre>118 140 162 184 21 22 44 66 88 100 132 154 176 198</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200	122 144 166 188 26 48 70 92 114 136 158 180 	124 146 168 190 28 50 72 94 116 138 160 182	126 148 170 192 30 52 74 96 118 140 162 184	 128 150 172 194 32 54 76 98 120 142 164 186 	130 152 174 196 34 56 78 100 122 144 166 188	132 154 176 198 36 58 80 102 124 146 168	134 156 178 200 38 60 82 104 126 148 170		
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 Ven n	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 umber	118 140 162 184 22 44 66 88 110 132 154 176 198 ℃S [≤	120 142 164 186 24 46 68 90 112 134 156 178 200 200]	122 144 166 188 26 48 70 92 114 136 158 180 as ge	124 146 168 190 28 50 72 94 116 138 160 182 nerat	126 148 170 192 30 52 74 96 118 140 162 184 ed by	128 150 172 194 32 54 76 98 120 142 164 186 Sub-	130 152 174 196 34 56 78 100 122 144 166 188 Sets	132 154 176 198 36 58 80 102 124 146 168 190	134 156 178 200 38 60 82 104 126 148 170 192	mbined =	(11)
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 ven n }, {	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 .umber 5 + P ₁	<pre>118 140 162 184 21 22 44 66 88 100 132 154 176 198 >s [≤ }, {7</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 + P ₁ }	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11	124 146 168 190 28 50 72 94 116 138 160 182 nerat + P _i },	126 148 170 192 30 52 74 96 118 140 162 184 ed by , {13	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P _i }	130 152 174 196 34 56 78 100 122 144 166 188 Sets and {	132 154 176 198 36 58 80 102 124 146 168 190	134 156 178 200 38 60 82 104 126 148 170 192 i } cor	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 ven n }, {	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 umber 5 + P ₁ 6	<pre>118 140 162 184 21 22 44 66 88 100 132 154 176 198 **s [≤ }, {7 8</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 + P _i } 10	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11 12	124 146 168 190 28 50 72 94 116 138 160 182 nerat + P _i }, 14	126 148 170 192 30 52 74 96 118 140 162 184 (13) (13)	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P _i } 18	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22	134 156 178 200 38 60 82 104 126 148 170 192 i} cor 24	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 yen n }, { 4 26	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 .umber 5 + P ₁ 6 28	118 140 162 184 21 44 66 88 110 132 154 176 198 30	120 142 164 186 24 46 68 90 112 134 156 178 200 + P _i } 10 32	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11 12 34	124 146 168 190 28 50 72 94 116 138 160 182 nerat + P _i }, 14 36	126 148 170 192 30 52 74 96 118 140 162 184 (13) (13) 16 38	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P ₁ } 18 40	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44	134 156 178 200 38 60 82 104 126 148 170 192 i } cor 24 46	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 Ven n }, { 4 26 48	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 umber 5 + P ₁ 6 28 50	<pre>118 140 162 184 2i = 22 44 66 88 110 132 154 176 198 30 52</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 200] + P ₁ } 10 32 54	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11 12 34 56	124 146 168 190 28 50 72 94 116 138 160 182 nerat + P _i }, 14 36 58	126 148 170 192 30 52 74 96 118 140 162 184 (13) (13) (13) 16 38 60	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P _i } 18 40 62	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42 64	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44 66	134 156 178 200 38 60 82 104 126 148 170 192 i } cor 24 46 68	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 yen n }, { 4 26 48 70	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 .umber 5 + P ₁ 6 28	<pre>118 140 162 184 2i = 22 44 66 88 100 132 154 176 198 30 52 74</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 200] + P _i } 10 32 54 76	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11 12 34 56 78	124 146 168 190 28 50 72 94 116 138 160 182 nerat + P ₁ }, 14 36 58 80	126 148 170 192 30 52 74 96 118 140 162 184 ed by (13) 16 38 60 82	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P ₁ } 18 40 62 84	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42 64 86	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44 66 88	134 156 178 200 38 60 82 104 126 148 170 192 i } cor 24 46 68 90	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 Ven n }, { 4 26 48 70 92	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 196 196 196 28 50 72 94	<pre>118 140 162 184 2i = 22 44 66 88 110 132 154 176 198 30 52</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 + P ₁ } 10 32 54 76 98	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11 12 34 56	124 146 168 190 28 50 72 94 116 138 160 182 nerat +P ₁ }, 14 36 58 80 102	126 148 170 192 30 52 74 96 118 140 162 184 ed by (13) (13) 16 38 60 82 104	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P ₁ } 18 40 62 84 106	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42 64 86 108	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44 66	134 156 178 200 38 60 82 104 126 148 170 192 i} cor 24 46 68 90 112	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 yen n }, { 4 26 48 70	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 umber 5 + P ₁ 6 28 50 72	<pre>118 140 162 184 2i = 22 44 66 88 100 132 154 176 198 30 52 74</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 200] + P ₁ } 10 32 54 76 98 120	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11 12 34 56 78	124 146 168 190 28 50 72 94 116 138 160 182 nerat + P ₁ }, 14 36 58 80	126 148 170 192 30 52 74 96 118 140 162 184 ed by (13) 16 38 60 82	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P ₁ } 18 40 62 84	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42 64 86	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44 66 88 110 132	134 156 178 200 38 60 82 104 126 148 170 192 i } cor 24 46 68 90	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 Ven n }, { 4 26 48 70 92	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 196 196 196 28 50 72 94	<pre>118 140 162 184 21 22 44 66 88 100 132 154 176 198 30 52 74 96</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 + P ₁ } 10 32 54 76 98	122 144 166 188 26 48 70 92 114 136 158 180 411 12 34 56 78 100	124 146 168 190 28 50 72 94 116 138 160 182 nerat +P ₁ }, 14 36 58 80 102	126 148 170 192 30 52 74 96 118 140 162 184 ed by (13) (13) 16 38 60 82 104	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P ₁ } 18 40 62 84 106	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42 64 86 108	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44 66 88 110	134 156 178 200 38 60 82 104 126 148 170 192 i} cor 24 46 68 90 112	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 yen n }, { 4 26 48 70 92 114	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 .umber 5 + P ₁ 6 28 50 72 94 116	<pre>118 140 162 184 21 22 44 66 88 100 132 154 176 198 30 52 74 96 118</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 200] + P ₁ } 10 32 54 76 98 120	122 144 166 188 26 48 70 92 114 136 158 180 411 12 34 56 78 100 122	124 146 168 190 28 50 72 94 116 138 160 182 nerat + P _i }, 14 36 58 80 102 124	126 148 170 192 30 52 74 96 118 140 162 184 (13) (13) (13) (13) (14) 20 (12) (13) (14) (12) (13) (12) (13) (13) (13) (13) (14) (14) (15) (15) (15) (15) (15) (15) (15) (15	128 150 172 194 32 54 76 98 120 142 164 186 Sub- + P ₁ } 18 40 62 84 106 128	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42 64 86 108 130	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44 66 88 110 132	134 156 178 200 38 60 82 104 126 148 170 192 i} cor 24 46 68 90 112 134	mbined =	
The ex	136 158 180 et {1 18 40 62 84 106 128 150 172 194 Ven n }, { 4 26 48 70 92 114 136	138 160 182 7 + F 20 42 64 86 108 130 152 174 196 .umber 5 + Pi 6 28 50 72 94 116 138	<pre>118 140 162 184 2i = 22 44 66 88 100 132 154 176 198 30 52 74 96 118 140</pre>	120 142 164 186 24 46 68 90 112 134 156 178 200 200] + P _i } 10 32 54 76 98 120 142	122 144 166 188 26 48 70 92 114 136 158 180 as ge , {11 12 34 56 78 100 122 144	124 146 168 190 28 50 72 94 116 138 160 182 ***********************************	126 148 170 192 30 52 74 96 118 140 162 184 ed by , {13 16 38 60 82 104 126 148	<pre>128 150 172 194 32 54 76 98 120 142 164 186 Sub- + Pi} 18 40 62 84 106 128 150</pre>	130 152 174 196 34 56 78 100 122 144 166 188 Sets and { 20 42 64 86 108 130 152	132 154 176 198 36 58 80 102 124 146 168 190 17+ P 22 44 66 88 110 132 154	134 156 178 200 38 60 82 104 126 148 170 192 i } cor 24 46 68 90 112 134 156	mbined =	

Each of the individual single sub-set tables is identical to its predecessor except that in each successive table the number in each position is increased by the amount that the index has increased. The effect is that the numbers in the sequence of tables move continuously to the left and upward in

International Journal of Scientific and Innovative Mathematical Research (IJSIMR) Page 48

the tables while the table positions of unsatisfactory even numbers [ones in bold face because they are in the positions where there is no number in P_{i} , where the corresponding number in the sequence of all odd numbers is a non-prime] remain unmoved.

The original sequence, that of sub-set $\{3 + P_i\}$, sets the problem. It exhibits many even numbers as a sum of two primes, that is as satisfactory numbers, and it also exhibits many gaps, uninterrupted sequences of unsatisfactory even numbers. The length of a gap, G, is its number of unsatisfactory numbers in uninterrupted sequence. In general, then, what is required to insure that every even number is generated as the sum of two primes? What is required to clear all of the gaps?

As the numbers in the individual single sub-set tables move to the left and upward with the increases in the sub-set index, I, whenever a number from a gap moves onto the position of a prime [a nonbold face position] the number is expressed as a sum of two primes, a satisfactory even number, and the gap is reduced to that extent. Not more than G such events would be needed to clear the gap. Each valid sub-set that acts after the original $\{3 + P_{i}\}$ is a candidate to provide such an event for each gap and whether it succeeds or not it moves the gap nearer to the beginning of the table.

From this point of view, clearing a gap of length G requires that the original odd number sequence, equation (1), exhibit at least G primes ahead of the gap. This requirement is conservative because sometimes the same position in a table clears more than one element of a gap, that is one or more of the G primes ahead of the gap in the table may sometimes act more than once on the same gap. [For example, for the G = 3 gap 94 96 98 per sub-set $\{3 + P_{i}\}$, equation (2), it turns out that the prime 92 actually clears two elements in the gap, the 94 and 96 [see equations (3), (4), (5), and And, for another example, equation (8) versus equation (10) where 122 and 126 are cleared (6)].simultaneously.] That type of help in clearing gaps is ignored in the present analysis so as to arrive at a worst case.

Another effect may further increase the above requirement. It can happen that a number in a gap could move onto the position of a prime in a sub-set having a non-prime index; or rather, the number in the gap could move more than one number to the left at one time because of the index skipping an intervening invalid sub-set [see equations (5) and (7) between which an invalid sub-set is skipped and likewise equations (9) and (11)]. However viewed, such an event could "waste" a prime, result in failure to benefit from it by the conversion of a number from unsatisfactory to satisfactory, meaning that there may need to be more than G available primes preceding the gap. [For example, the **98** in sub-set $\{7 + P_{i}\}$, equation (5), is in position to move to the left onto a position [non-bold face] that would clear it; however, the next sub-set is $\{9 + P_i\}$, an invalid sub-set that is skipped over because its non-prime index prevents clearing any elements. The next position after that is in bold face again and unable to clear an element. The net effect is that the potentially clearing position is missed. "wasted".]

How many more available primes might be needed to account for and offset this effect? A conservative estimate of the amount that G should be increased would be to multiply G by the ratio of the number of non-primes preceding the gap to the number of primes preceding the gap, that is increase G to $G \cdot [1 + that ratio]$. That is equivalent to assuming that every invalid sub-set wastes a prime. Sometimes the "wasted prime" position is occupied by a bold face number or sometimes the number in position to move there is already cleared so that, either way nothing is really lost. [For example, the **142** in sub-set $\{7 + P_i\}$, equation (5).] An adjustment to take account of that help in clearing gaps is ignored in the present analysis so as to arrive at a worst case.

5. STEP 4

The required number of available primes preceding the gap now becomes as follows.

Where:
$$n \equiv$$
 the first number of a gap, the gap's beginning. (13)
 $\pi(n) \equiv$ the number of primes that are $\leq n$.
 $R(n) \equiv$ the required number of primes ahead of the gap that
begins at number "n" needed to clear that gap.
Then:

$$R(n) = G \cdot [1 + [n-\pi(n)]/\pi(n)]$$

=
$$G \cdot [n/\pi(n)]$$

International Journal of Scientific and Innovative Mathematical Research (IJSIMR) Page 49

6. STEP 5

Why there are always sufficient primes preceding each gap.

The subject of the distribution of primes has been studied in depth. Chapter 4, "How are the Prime Numbers Distributed?" of Reference [1] summarizes the history and results of those studies and presents a number of related proofs. The results fall into two categories for the present purposes: section I of the chapter, which treats the development of the Prime Number Theorem, that is expressions for the number of primes in designated intervals, and section II of the chapter, which treats gaps between primes. That referenced work is the source of the following data.

The Prime Number Theorem, the most fundamental theorem of prime numbers is as follows.

 $\pi(n) \equiv$ the prime counting function

(14)

= the number of primes \leq n, an integer

 \approx n/_{Ln(n)} the approximation improving as n increases

That approximation is low by 5.78% for $n = 10^8$ improving to low by 2.79% for $n = 10^{16}$. A better approximation is given by a function called the logarithmic interval as follows.

$$\pi(n) \approx \operatorname{Li}(n) = \int_{2}^{n} dx / \operatorname{Ln}(x)$$
(15)

The logarithmic interval approximation to $\pi(n)$ is high by only 0.013% for $n = 10^8$ and by only 0.000,000,5% for $n = 10^{16}$.

Even more accurate is the Riemann function, too involved to be worth specifying here, which for $n = 10^8$ differs from the correct value by only 0.0017% and for $n = 10^{16}$ by 0.000,000,1%.

Substituting equation (9) into equation (8) the following is obtained.

$$R(n) = G \cdot \left[\frac{n}{\pi(n)}\right]$$

$$= G \cdot n \cdot \left[\frac{\ln(n)}{n}\right]$$

$$= G \cdot \ln(n)$$
(16)

The issue is, of course, how does R(n), the number of preceding primes required, compare with $\pi(n)$, the number of preceding primes actually available? That is as follows.

$$\pi(n) /_{R(n)} = \frac{n /_{Ln(n)}}{G \cdot Ln(n)}$$

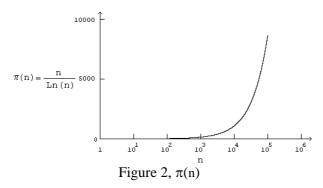
$$= \frac{n /_{G} \cdot [Ln(n)]^{2}}{G \cdot Ln(n)}$$
(17)

Fig. 1, below lists some values for that function using values for *G* extrapolated from the percent deviation of the logarithmic interval from the exact count of $\pi(n)$ presented above. From hundreds to thousands to even far greater multiples of the required number of primes preceding the gaps are actually available for clearing them.

n	G	$\pi(n)/R(n)$	G∕n %
108	103.0	295	0.001
109	10 ^{3.6}	585	0.0004
1010	104.3	945	0.0002
10 ¹¹	104.9	1962	0.00008
10 ¹²	105.5	4142	0.00003
1013	106.1	8865	0.00001
10 ¹⁴	106.8	15251	0.00006
10 ¹⁵	107.4	33372	0.00003
1016	108.0	73676	0.000001

The formulations of various accuracies for evaluating $\pi(n)$ cited on the previous page: n/Ln(n), the logarithmic interval and the Riemann function, cannot be used to find individual prime numbers. Precise, not limited accuracy, is required for that. However, the formulations do indicate that the prime numbers are well distributed over the range of all numbers, that dense concentrations and large gaps cannot occur.

The function of equation (9) is smooth as shown in Fig. 2, below, and while the function only approximates the actual values of $\pi(n)$, that approximation is fairly close over most of the range. The same can be said, and more emphatically, of the more accurate functions cited for which the approximation is better. Gaps that are large relative to the location in the sequence of numbers, that is other than small values of $G/\pi(n)$ simply cannot occur. The precise condition, $\pi(n) \ge R(n) = G \cdot Ln(n)$, is greatly exceeded.



For example, at $n = 10^8$, the deviation of $\pi(n)$ from the exact number of primes up to that n is 0.0017%, which is a number range of 1,700 out of 100,000,000. The largest gap in that neighborhood could not reasonably exceed twice that 3,400, and certainly not 10 times it. The number of primes available up to that point is 5,761,455. That is far more than needed to eliminate the effect of the gap. [The largest gap that has ever been found is a sequence of 653 non-primes following the prime 11,000,001,446,613,353 at which the number of preceding primes available is more than 279,238,341,033,925 (the value for 10^{16}).]

7. IN SUMMATION

a. - All of the prime numbers other than 2 are odd, 2 being the only even prime number. Further, the even number 4 = 2 + 2.

b. - The sum of any two of the odd prime numbers is always an even number.

c. - All combinations (the combinations in pairs may include the same number twice) of the odd numbers ≥ 3 [whether prime or not] summed in pairs produces all of the even numbers ≥ 6 .

d. - While just the prime odd numbers in sequence is a sequence with gaps as compared to that of all of the odd numbers; nevertheless, all combinations of the odd prime numbers ≥ 3 summed in pairs produces all of the even numbers provided that there are enough primes preceding the gaps.

e. - That requirement is that $\pi(n) \ge R(n) = G \cdot Ln(n)$ where *n* is the first number in the gap, $\pi(n)$ is the number of primes less than or equal to *n*, R(n) is the number of preceding primes needed to assure clearance of the gap, and *G* is the number of sequential non-primes in the gap. This requirement is comprehensively satisfied by all of the prime numbers and gaps because of the sufficiently smooth nature of $\pi(n)$.

Which proves the conjecture.

REFERENCES

[1] Ribenboim, P., *The Book of Prime Number Records, Second Edition*, Springer-Verlag, 1989, Library of Congress catalog # 89-21675.

AUTHOR'S BIOGRAPHY



A graduate of the United States Military Academy at West Point and post-graduate of Stanford University Roger Ellman left the military and pursued his primary interest in the origin of the universe and its cosmological physics. That led to the study of gravitation, various astronomical anomalies and philosophy. He is the author of a number of scientific papers on physics and astrophysics and several books.