International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 1, January 2015, PP 45-54
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Harvesting of Both Commensal-Host Species Pair at a Constant Rate -A Numerical Approach

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Abstract: *In this paper, an analytical investigation on two species commensal-host model is investigated. In this case, both the species are considered with limited resources and harvested at a constant rate. The commensal species (S_1), in spite of the limitation of its natural resources flourishes drawing strength from the host species (S_2).This model is characterized by a pair of first order non-linear coupled differential equations. In all, nine equilibrium points of the model are identified and their stability criteria are discussed. Solutions for the linearized perturbed equations are found and the results are illustrated. The growth rate equations are solved numerically by employing Runge-Kutta fourth order method. Further, some threshold results are illustrated.*

Keywords: *Mathematical model, Commensalism, Species, Differential equations, Runge-Kutta method, Stability analysis.*

1. INTRODUCTION

Lotka[12] and Volterra[21] initiated mathematical studies of eco-systems in general, more particularly problems related to growth and decay of fisheries. The ecological symbiosis of living species can be broadly classified as Prey- Predation, Competition, Mutualism, Commensalism, Ammensalism, and Mutation and so on. Meyer [13], Kapok [6, 7] and several others dealt at length in their treatises. The general concepts of mathematical modeling of ecosystems were established strongly by many mathematicians. The stability of biological communities in nature was discussed by Svirezher and D.O.Logofet [19]. Competition between two and three species with limited and unlimited resources was studied earlier by Srinivas [20] .The later work was followed by Lakshminarayan and PattabhiRamacharyulu [8, 9, 10] with their investigations on Prey-Predator ecological models with partial cover for the prey and alternate food for the predator and also models with harvesting. A Prey-Predator model with a variable cover for the prey and alternate food for the predator was studied by Lakshminarayan and Paparao[11].Later PattabhiRamacharyulu et.al [2],Archanareddy[3]and Rama Sarma[4,5] discussed the stability of species in competition. The mutualism was considered by Ravindra Reddy [17]. Following this, PattabhiRmacharyulu & Phanikumar,et.al investigated the stability of species in commensalism[14,15,16,18]. K.V.L.N Aacharyulu [1] and PattabhiRamacharyulu N.Ch. explicated various important cases of Ammensalism and obtained some fruitful results.

The present investigation is devoted to the analytical study of commensalism between two species. A two species Commensalism is an ecological relationship between two species where one species (S_1) derives a benefit from the other (S_2) which does not get affected by it: S_1 may be referred as the commensal species while S_2 is the host. Some examples are Cattle Egret, Anemonetish, and Barnacles etc.

The host species (S_2) supports the commensal species (S_1) which has a natural growth rate in spite of a support other than from S_1.The commensal species (S_1), in spite of the limitation of its natural resources flourishes drawing strength from the host species (S_2).The model is characterized by a coupled pair of first order non-linear differential equations. The system has **nine** co-existent equilibrium states (E_1)-(E_9) resulting from $\frac{dN_1}{dt} = 0$; $\frac{dN_2}{dt} = 0$ and these are classified into two categories A and B by depending and in -depending on Harvesting rate. In all Nine Equilibrium states The Equilibrium state E_1 is only stable.

2. BASIC EQUATIONS

Notation Adopted:

N_1, N_2 : The population rates of the commensal (S_1) and host (S_2) species respectively at time t

$K_i = \frac{a_i}{a_{ii}}$: Carrying capacities of S_i, i = 1, 2 (these parameters characterize the amount of resources available for the consumption exclusively for the two species.)

$C = \frac{a_{12}}{a_{11}}$: Commensalism coefficient.

Further both the variables N_1 and N_2 are non-negative and the model parameters a_1, a_2, a_{11}, a_{22}, and a_{12} are assumed to be non-negative constants.

Employing the above terminology, the model equations for a two species commensaling system are given by the following system of non- linear differential equations.

Equations for the growth rate of the commensal species (S_1) and host species (S_2) can be written as follows:

$$\frac{dN_1}{dt} = a_{11}\, [\, K_1 N_1 - N_1^2 + C\, N_1 N_2 - H_1]$$

$$\frac{dN_2}{dt} = a_{22}\, [\, K_2 N_2 - N_2^2 - H_2\,] \tag{1}$$

3. MATHEMATICAL ANALYSIS

3.1 Existence of Equilibrium Points

In order to find out the equilibrium points of the system (1), we set $\dot{N}_1 = \dot{N}_2 = 0$ where

$$\dot{N}_i = \frac{dN_i}{dt},\ i = 1, 2. \tag{2}$$

We can find that there are nine non-negative equilibrium points and these are classified into two categories A and B.

A) When The Harvesting Rates are Interdependent

$$\text{When } H_1 < \frac{1}{4}\left[K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)\right]^2 ; H_2 < \frac{K_2^{\,2}}{4} \tag{A.1}$$

E_1:

$$\overline{N_1} = \left(K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)\right) - \frac{H_1}{K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)} ;\quad \overline{N_2} = K_2 - \frac{H_2}{K_2} \tag{3}$$

E₂: $$\overline{N_1} = \frac{H_1}{K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)}\;;\; \overline{N_2} = K_2 - \frac{H_2}{K_2} \qquad (4)$$

The above two states exist only when $K_2^{\,2} > H_2$ and $\left[K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)\right]^2 > H_1$

B. When The Harvesting Rates are Not Interdependent

(B.1) When $H_1 > \frac{1}{4}\left[K_1 + \frac{3CK_2}{4}\right]^2 ; H_2 < \frac{K_2^{\,2}}{4}$ (B.1)

E₃: $$\overline{N_1} = \frac{K_1 + C\ \ K_2 - \frac{H_2}{K_2}}{2}\;;\; \overline{N_2} = K_2 - \frac{H_2}{K_2} \qquad (5)$$

This exists only when $K_2^{\,2} > H_2$

(B.2) When $H_1 < \frac{1}{4}\left[K_1 + \frac{CK_2}{4}\right]^2 ; H_2 < \frac{K_2{}^2}{4}$ (B.2)

E₄: $$\overline{N_1} = K_1 + \frac{CH_2}{K_2} - \frac{H_1}{K_1 + \frac{CH_2}{K_2}}\;;\; \overline{N_2} = \frac{H_2}{K_2} \qquad (6)$$

This happens only when $K_1 + \frac{CH_2}{K_2}\ ^2 > H_1$

E₅: $$\overline{N_1} = \frac{H_1}{K_1 + \frac{CH_2}{K_2}}\;;\; \overline{N_2} = \frac{H_2}{K_2} \qquad (7)$$

E₆: $$\overline{N_1} = \frac{K_1 + \frac{CH_2}{K_2}}{2}\;;\; \overline{N_2} = \frac{H_2}{K_2} \qquad (8)$$

(B.3) When $H_1 < \frac{1}{4}\left[K_1 + \frac{CK_2}{2}\right]^2 ; H_2 = \frac{K_2{}^2}{4}$ (B.3)

E₇: $$\overline{N_1} = K_1 + \frac{CK_2}{2} - \frac{H_1}{K_1 + \frac{CK_2}{2}}\;;\; \overline{N_2} = \frac{K_2}{2} \qquad (9)$$

This exists only when $K_1 + \frac{CK_2}{2}\ ^2 > H_1$

E₈: $$\overline{N_1} = \frac{H_1}{K_1 + \frac{CK_2}{2}}\;;\; \overline{N_2} = \frac{K_2}{2} \qquad (10)$$

(B.4) When $H_1 = \frac{1}{4}\left[K_1 + \frac{CK_2}{2}\right]^2 ; H_2 = \frac{K_2{}^2}{4}$ (B.4)

E₉: $$\overline{N_1} = \frac{1}{2}\left[K_1 + \frac{CK_2}{2}\right];\; \overline{N_2} = \frac{K_2}{2} \qquad (11)$$

3.2 Stability Analysis

The local stability of each equilibrium point can be studied by computing the corresponding variational matrix,

$$A = \begin{bmatrix} a_{11}(K_1 - 2\bar{N}_1 + C\bar{N}_2) & a_{11}C\bar{N}_1 \\ 0 & a_{22}(K_2 - 2\bar{N}_2) \end{bmatrix}$$

From variational matrix analysis, the local stability of equilibria can be concluded on existing equilibrium points as below.

1. E_1 is stable. 2. E_2 is unstable. 3. E_3 is unstable. 4. E_4 is unstable.5. E_5 is unstable.

6. E_6 is unstable. 7. E_7 is unstable. 8. E_8 is unstable. 9. E_9 is unstable.

4. A Numerical Solution of the Basic Non-Linear Coupled Differential Equations: The Variation of N_1 and N_2 Verses Time T

The variation of N_1 and N_2 verses time t in the interval [0, 10] is computed numerically by employing Runge-Kutta technique for a wide range of values of the characterizing parameters a_1, a_2; a_{11},a_{22}; a_{12},H_1,H_2as shown in Table-1.For this, MATLAB has been used and the results are illustrated from Fig.1 to Fig.6(The interesting point between the two curves in the respective figures indicates the time instant(t*).

Table-1

S.No	a_1	a_{11}	a_{12}	a_2	a_{22}	H_1	H_2	N_{10}	N_{20}	The corresponding figure mentioned in this table indicates the variation of N_1 & N_2 vs. 't'
1	0.1	0.05	0.001	0.09	0.03	0.56	0.35	1.78	2.88	Fig.1
2	1	0.5	0.3	2	0.5	0.95	0.15	1	2.5	Fig.2
3	1	0.05	2	2	0.005	1.5	0.15	1	2.5	Fig.3
4	1	0.06	2	2	0.005	1.5	1.2	1	2.5	Fig.4
5	1	0.05	2	2	0.005	1	1.5	2	2.5	Fig.5
6	2	0.5	0.2	3	0.9	1.5	1.8	1	2	Fig.6

Case-1:

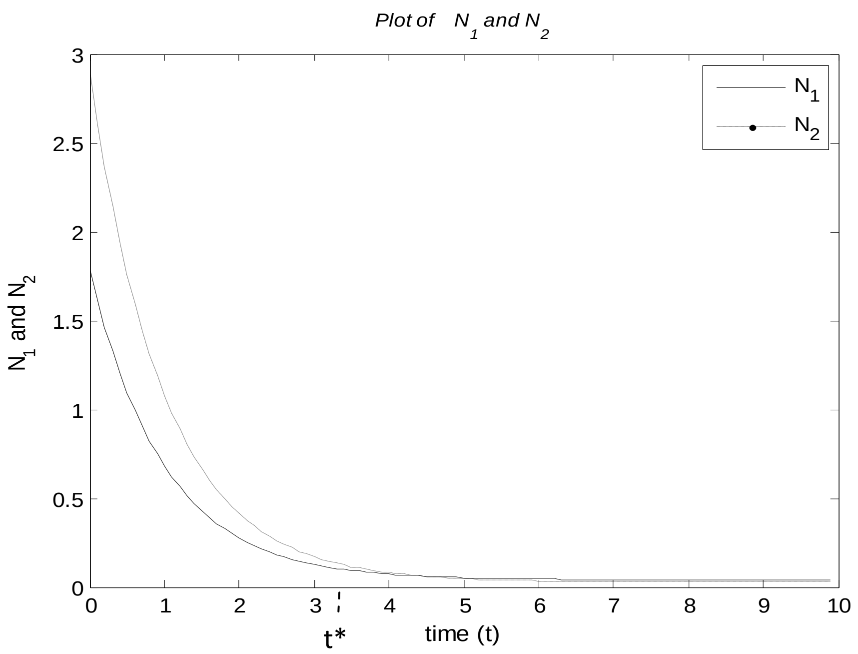

Fig .1. *Variation of N_1, N_2 vs. t for a_1=0.1, a_{11}= 0.05, a_{12}=0.001, a_2=0.09, a_{22}=0.03, H_1=0.56 H_2=0.35 N_{10}=1.78, N_{20}=2.88*

In this case the first species dominates over the second species initially. Both the species suffer a steep fall and after a time t*=3.8 both the species appear to be almost extinct with negligible growth rates.

Case-2:

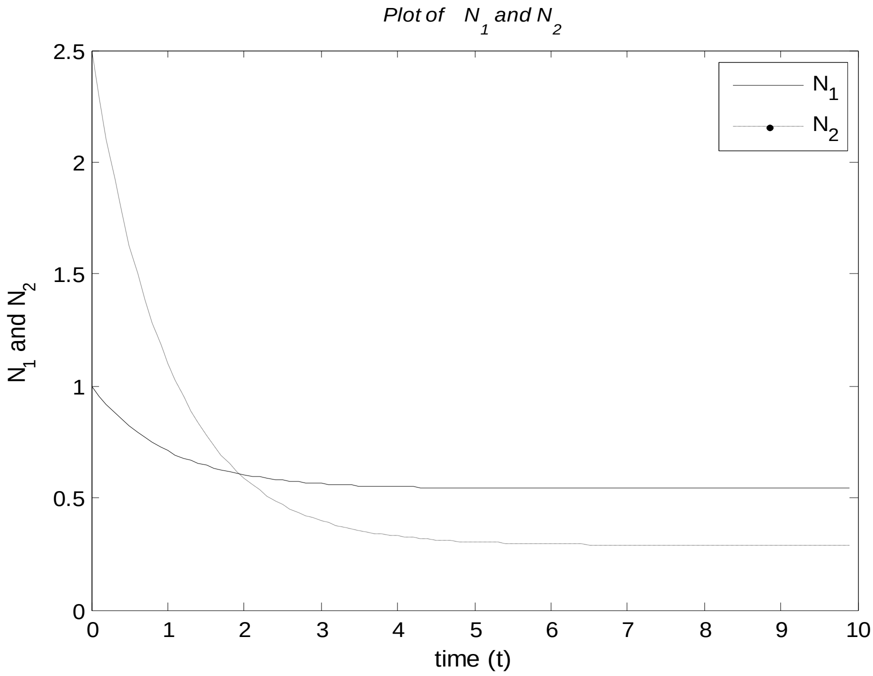

Fig .2. *Variation of N_1, N_2 vs. t for a_1=1, a_{11}= 0.5, a_{12}=0.3, a_2=2, a_{22}=0.5, H_1=0.95 H_2=0.15 N_{10}=1, N_{20}=2.5*

Initially it is noticed that a steady decrease in both the species. The second species suppress the first up to a time t*=1.9, after which the dominance is reversed. Further both the species maintain steady variation with low growth rates as seen in Fig.2.

Case-3:

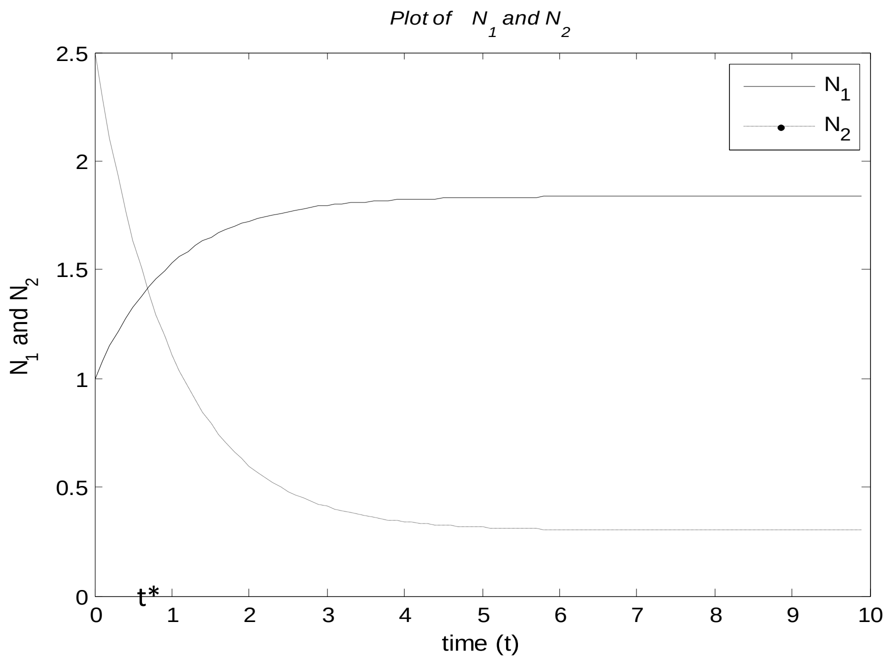

Fig.3. *Variation of N_1, N_2 vs. t for a_1=1, a_{11}= 0.05, a_{12}=2, a_2=2, a_{22}=0.005, H_1=1.5 H_2=0.15 N_{10}=1, N_{20}=2.5*

In this case initially the first species suppressed by the second species but after a time t*=0.8 the second species is suppressed. Further we see that the first species rises initially and later maintains a steady variation with no appreciable growth rate. Whereas the second species decreases initially and in course of time it is almost extinct as seen in Fig.3.

Case-4:

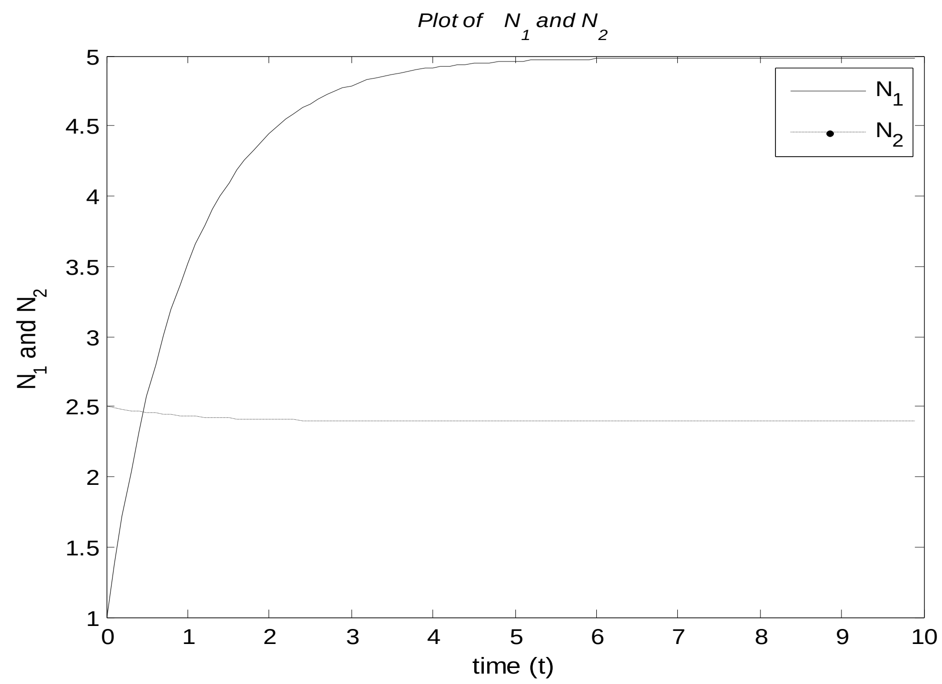

Fig.4. *Variation of N_1, N_2 vs. t for a_1=1, a_{11}= 0.05, a_{12}=2, a $_2$=2, a $_{22}$=0.005, H_1=1.5 H_2=1.2 N_{10}=1, N_{20}=2.5*

In this case the second species dominates the first species initially but after time t*=0.4 the first species dominates the second species .In course of time both the species co-exist with a steady variation and no appreciable growth as seen in Fig.4.

Case-5:

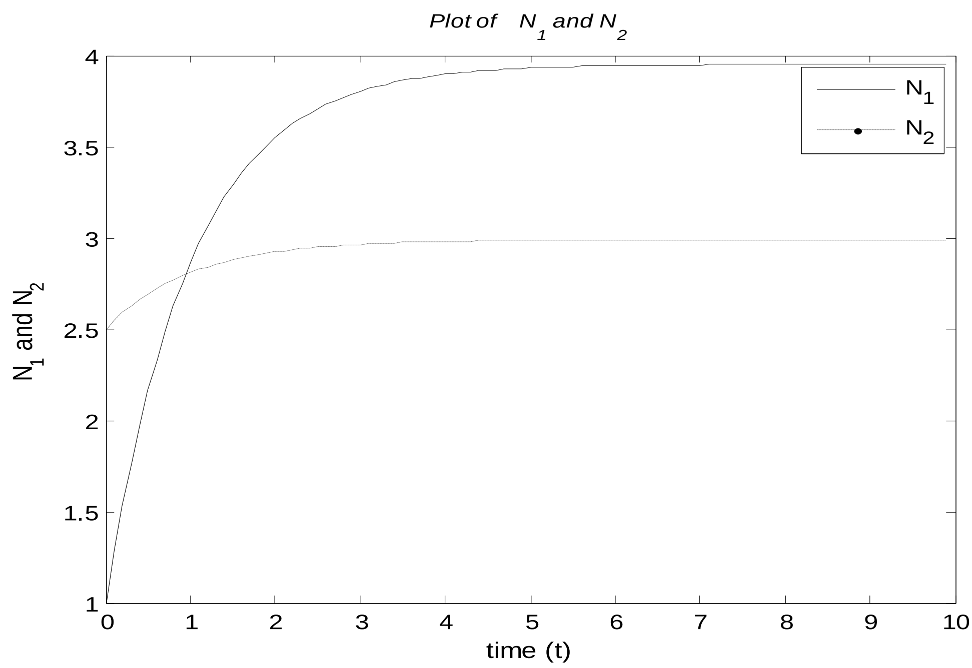

Fig.5. *Variation of N_1, N_2 vs. t for a_1=1, a_{11}= 0.05, a_{12}=2, a $_2$=2, a $_{22}$=0.005, H_1=1 H_2=1.5 N_{10}=1, N_{20}=2.5*

Initially the second species dominates the first species. The dominance reversal time t*=0.9 is shown in Fig.5.Further we observe that both the species are co-existing with a steady variation.

Case -6

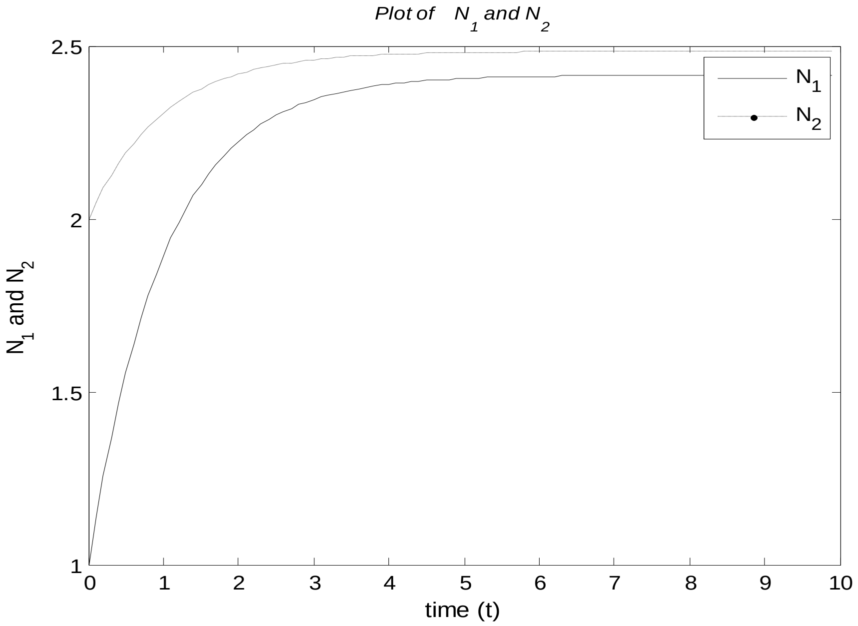

Fig.6. *Variation of N_1, N_2 vs. t for a_1=2, a_{11}= 0.5, a_{12}=0.2, a_2=3, a_{22}=0.9, H_1=1.5 H_2=1.8 N_{10}=1, N_{20}=2*

In this case the second species always dominates the first species. Further both the species are co-exist with no appreciable growth rate as shown in Fig.4.6.

5. Threshold (Or) Phase – Plane Diagram

The conditions $\frac{dN_1}{dt}=0$ and $\frac{dN_2}{dt}=0$ imply that neither N_1 nor N_2 changes its density .When we impose these conditions the basic equations give rise to hyperbola and straight line. At the points where $\frac{dN_1}{dt}=0, \frac{dN_2}{dt}=0$, the resulting curves divide the phase plane into nine regions (vide **Fig.7**).

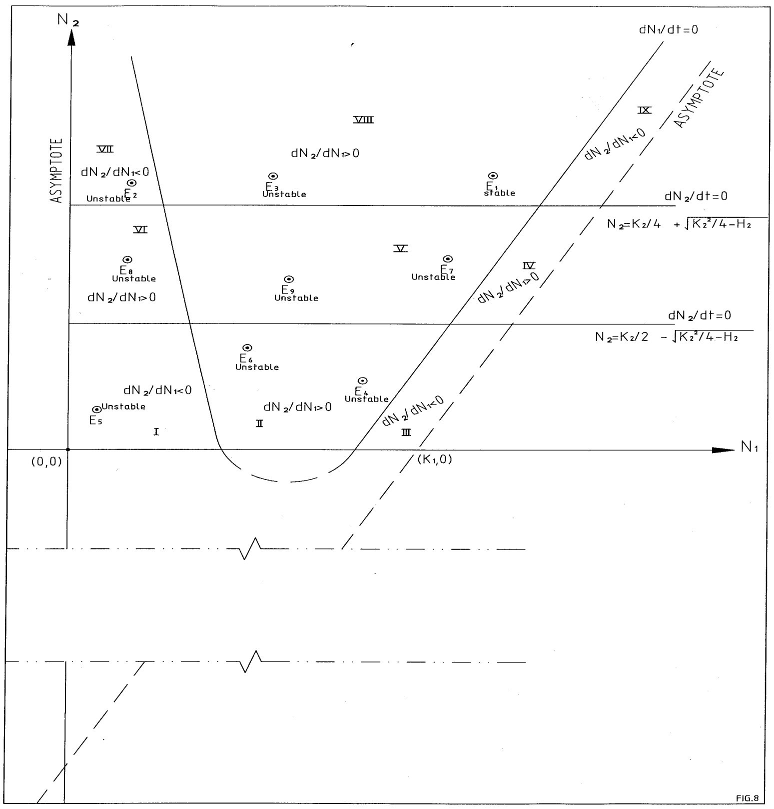

Fig.7. *Threshold regions*

The regions are classified as nine categories as

Region I : The commensal species N_1 flourish and the host species N_2 declines with time t.

Region II : Both the species N_1 and N_2 decline with time t.

Region III : The commensal species N_1 flourish and the host species N_2 declines with time t.

Region IV : Both the species N_1 and N_2 flourish with time t.

Region V : The commensal species N_1 declines and the host species N_2 flourish with time t.

Region VI : Both the species N_1 and N_2

Region VII : The commensal species N_1 flourish and the host species N_2 declines with time t.

Region VIII : Both the species N_1 and N_2 decline with time t.

Region IX : The commensal species N_1 flourish and the host species N_2 declines with time t.

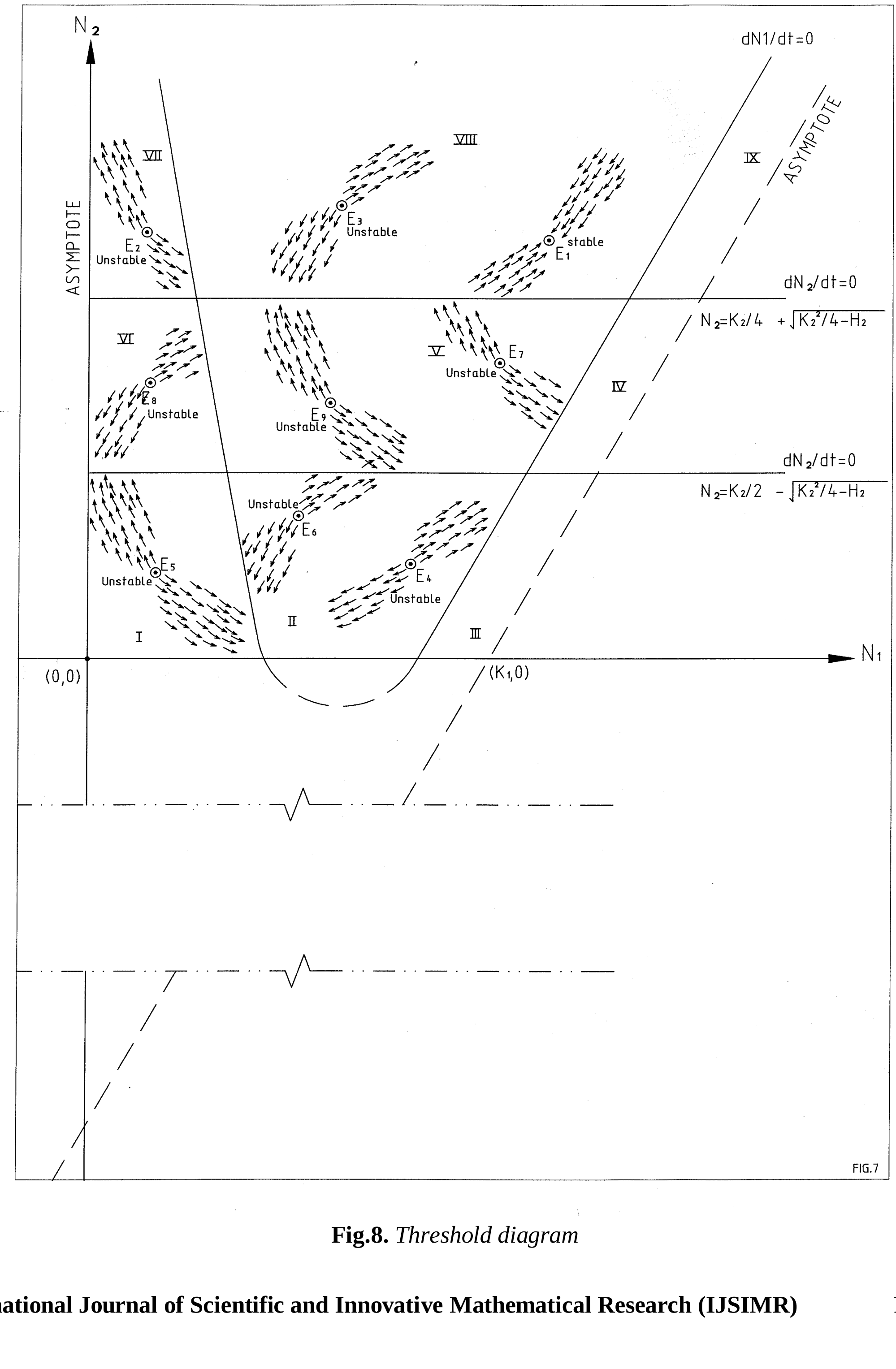

Fig.8. *Threshold diagram*

ACKNOWLEDGMENT

The authors are grateful to **Prof.N.Ch. Pattabhi Ramacharyulu**, Retd.Professor, Department of Mathematics, National Institute of Technology, Warangal, India, for his valuable guidance and constant encouragement.

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