On Groups with Chain Conditions on Subnormal Subgroups

Falih A.M. Aldosray  
Department of Mathematics  
Umm AlQura University,  
Makkah, Saudi Arabia  
fadosary@uqu.edu.sa

Abduallah A. Abduh  
Department of Mathematics  
Umm AlQura University  
Makkah, Saudi Arabia  
aaabduh@uqu.edu.sa

Abstract: Groups with chain Conditions on subnormal subgroups have been investigated by many authors. In this paper we give a necessarily and sufficient conditions under which a group $G$ satisfy the ascending or the descending chain conditions on subnormal subgroups.

Keywords: Prime Subgroups, Groups with chain Conditions on subnormal subgroups.

1. INTRODUCTION

Let $G$ be a group. A subgroup $P$ of $G$ is said to be a primesubgroup of $G$ if $P$ is normal in $G$ and $[A, B] \subseteq P$ with $A, B \triangleleft G$ implies that either $A \subseteq P$ or $B \subseteq P$. Here $[ , ]$ is the commutator. Following Scukin [11] we say that a group $G$ is prime if $[A, B] \neq 1$ whenever $A$ and $B$ are nontrivial normal subgroups of $G$, see also Dark [5]. Then $P$ is prime in $G$ if and only if $G/P$ is a prime group.

We define the soluble radical $\sigma(G)$ to be the product of all soluble normal subgroups of $G$. We say that $G$ is semisimple if $\sigma(G) = 1$. The terms of the derived and lower central series of $G$ are denoted $G^{(n)}$ and $\gamma_n(G)$ as in Robinson [8]. A prime subgroup $P$ of $G$ is said to be a minimal prime subgroup belonging to a normal subgroup $H$ if $P \supseteq H$ and if there is no prime subgroup between $H$ and $P$, see Kurata [6, p 205]. The radical $r(H)$ of a normal subgroup in $G$ is the intersection of all minimal prime subgroups belonging to $H$, see Kurata [6, p 206]. If $G$ is unclear we write this as $r(G)$. It follows that $r(H)$ is the intersection of all prime subgroups containing $H$, see Kurata [6, Proposition 1.13 p.207]. We write $r_G$ for $r_G(1)$, the intersection of all minimal prime subgroups of $G$.

We denote by $\text{Max} - \triangleleft$ the class of all groups satisfying the maximal condition on normal subgroups (often called Max-n), with similar definition for $\text{Min} - \triangleleft$. The classes of all groups satisfying the maximal (respectively minimal) condition on subnormal subgroups are denoted by Max-sn, and Min-sn, following Robinson[8], which is also our source for any other unexplained notation and determined by the corresponding chain condition, so that $G$ satisfies Min-sn and $G \in \text{Min} - sn$ are equivalent statement.

2. RESULT

Proposition 1: For all group $G$,

(a) $\sigma(G) \subseteq r_G$.

(b) If $G \in \text{Max} - \triangleleft$, then $\sigma(G) = r_G$.

Proof

(a) Let $H$ be a soluble normal subgroup of $G$. Then $H^{(n)} = 1$ for some $n \geq 0$. In particular $H^{(n)} \subseteq P$ for every prime subgroup $P$. Inductively we see that $H \subseteq P$, whence $\sigma(G) \subseteq r_G$. 

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(b) Let $R = r_G$ and suppose that $R$ not soluble. Let $C$ be the collection of all normal subgroups $N$ of $G$ such that $R^{(n)} \nsubseteq N$ for all integers $n \geq 0$. Then $C$ is non-empty since $1 \in C$. Hence $C$ has a maximal element say $p$. We claim that $P$ is prime. Suppose not, then there are normal subgroups $A, B$ of $G$ such that $A \not\subseteq P$ and $B \not\subseteq P$ but $[A, B] \subseteq P$. Therefore $AP, BP \subseteq C$. Hence $R^{(m)} \subseteq AP$ and $R^{(n)} \subseteq BP$ for some integers $m, n \geq 0$. Let $s = \max\{m, n\}$. Then $R^{(s+1)} \subseteq [AP, BP] = [AP, B][AP, P] = [A, B][P, B][A, P][P, P] \subseteq P$. Hence $AP \subseteq P$ or $BP \subseteq P$, which implies that $A \subseteq P$ or $B \subseteq P$, a contradiction. Hence $P$ is prime and $R \not\subseteq P$, another contradiction. Therefore $R$ is soluble so $R \subseteq \sigma(G)$. But $\sigma(G) \subseteq R$ by (a), so $R = \sigma(G)$ as claimed.

**Proposition 2**

(a) Let $G \in \text{Max}^\prec \sigma^3$. Then $\sigma(G)$ is soluble and $\sigma(G) \in \text{Max}$.

(b) Let $G \in \text{Min}^\prec \sigma^2$. Then $\sigma(G)$ is soluble and $\sigma(G) \in \text{Min}$.

**Proof:**

(a) Since in particular $G \in \text{Max}^\prec \sigma$ it follows that $S = \sigma(G)$ is the product of finitely many soluble normal subgroups, hence is soluble. Because $G \in \text{Max}^\prec \sigma^3$ we have $S \in \text{Max}^\prec \sigma^2$. Each derived factor $S^{\langle n \rangle}/S^{\langle n+1 \rangle}$ is abelian with $\text{Max}^\prec \sigma$, hence with $\text{Max}$. By E-closure of $\text{Max}$, we have $S \in \text{Max}^\prec \sigma$.

(b) By Theorem 5.49.1 or Robinson [8] p. 148 we have $\text{Min}^\prec \sigma^2 = \text{Min} - \text{sn}$. Now apply the analogous argument to part (a) with $\text{Max}$ replaced by $\text{Min}$.

**Proposition 3** Let $G$ be a group and $S$ be respectively the set of normal subgroups, subnormal subgroups, $n$-step subnormal subgroups of $G$. Suppose that $N_{i} \triangleleft G$ $(i = 1, \ldots, m)$ and $\cap_{i=1}^{m} N_{i} = 1$.

Let $S_{i} = \{HN_{i} : H \in S \} \in \text{Max} - S_{i}$ (respectively $\text{Min} - S_{i}$) for all $i$,
then $G \in \text{Max} - S$ (respectively $\text{Min} - S$).

**Proof:** This is equivalent to $R_{0}$-closure of these classes, see Robinson[8] Corollary to Lemma 1.48, p.39.

**Proposition 4** A group $G$ is a sub-direct product of a family of groups $\{ G_{a} \}_{a \in A}$ if and only if for each $a \in A$ there is a surjective homomorphism $g_{a} : G \rightarrow G_{a}$ such that $\bigcap_{a \in A} \ker g_{a} = 1$.

**Proof:** This is standard: compare Cohn [3, p.99]

**Corollary 5** Let $G$ be a group and let $G_{a} \big/ G_{a}$ be a family of normal subgroups of $G$.

If $\bigcap_{a \in A} G_{a} = 1$, then $G$ is a sub-direct product of the family of groups $\{ G \big/ G_{a} \}_{a \in A}$.

**Proposition 6**

(a) $G$ is semi-simple with $\text{Max}^\prec n \sigma^1$ $(n \geq 1)$ (respectively $\text{Max}^\prec \text{sn}$) if and only if $G$ is a sub-direct product of a finite number of prime groups satisfying $\text{Max}^\prec n \sigma^1$ $(n \geq 1)$ (respectively $\text{Max}^\prec \text{sn}$).

(b) $G$ is semi-simple with $\text{Min}^\prec n \sigma^1$ $(n \geq 1)$ (respectively $\text{Min}^\prec \text{sn}$) if and only if $G$ is a sub-direct product of a finite number of prime groups satisfying $\text{Min}^\prec n \sigma^1$ $(n \geq 1)$ (respectively $\text{Min}^\prec \text{sn}$).
On Groups with Chain Conditions on Subnormal Subgroups

Proof: (a) Let \( G \) be semisimple with \( \text{Max} - \sigma^n(n \geq 1) \) (respectively \( \text{Max} - sn \)). Then \( \sigma(G) = 1 \). By Kurata [3] Proposition 4p 214 we have \( r_G = \bigcap_{i=1}^{m} P_i \) where the \( P_i \) are minimal prime subgroups of \( G \). But by Proposition 1(b) \( \sigma(G) = r_G \), so \( \sigma(G) = 1 \). Since \( P_i \) is a prime subgroup the quotient \( G/P_i \) is prime, and by Q-closure it lies in \( \text{Max} - \sigma^n(n \geq 1) \) (respectively \( \text{Max} - sn \)). By Corollary 5 \( G \) is a subdirect product of prime groups satisfying \( \text{Max} - \sigma^n \) (respectively \( \text{Max} - sn \)).

To prove the converse suppose that \( G \) is a subdirect product of finitely many prime groups \( G_i \) where \( i = 1, \ldots, m \) and each \( G_i \) satisfies \( \text{Max} - \sigma^n \) (respectively \( \text{Max} - sn \)). Let \( g_i : G \to G_i \) be the homomorphism of Proposition 4. For each \( I \) we have \( G/\ker g_i \cong G_i \), and \( G_i \) is prime. So \( \ker g_i \) is a prime subgroup of \( G \). Thus \( r_G \subseteq \ker g_i \) for all \( i \), so \( r_G = 1 \). By proposition 1(a) also \( \sigma(G) = 1 \), so \( G \) is semisimple. That \( G \in \text{Max} - \sigma^n \) (respectively \( \text{Max} - sn \)) follows from Proposition 3.

(b) Let \( G \) be semisimple with \( \text{Min} - \sigma^n(n \geq 1) \) (respectively \( \text{Min} - sn \)). Then \( G \) has only a finite number of minimal normal subgroups where \( i = 1, \ldots, r \). Let \( P_i \) be a normal subgroup of \( G \) that is maximal with respect to not containing \( M_j \). We claim that \( P_i \) is a prime subgroup of \( G \). If not there exist normal subgroups \( A, B \) of \( G \) such that \( A \subseteq P_i \), \( B \subseteq P_i \), but \([A,B] \subseteq P_i \). Now \( P_i \subset AP_i \) and \( P_i \subset BP_i \), so by the choice of \( P_i \) we have \( AP_i \supseteq M_i \) and \( BP_i \supseteq M_i \). Therefore \( \gamma_2 M_i \subseteq [AP_i, BP_i] \subseteq P_i \), but \( \gamma_2 M_i \neq 1 \) since \( G \) is semi-simple so \( \gamma_2 M_i = M_i \subseteq P_i \). Therefore \( P_i \) is a prime subgroup of \( G \) and \( G/P_i \) is a prime group. If \( \bigcap_{i=1}^{m} P_i \neq 1 \), then this intersection contains some minimal subgroup \( M_j \). But \( M_j \not\subset P_i \), a contradiction. Therefore \( \bigcap_{i=1}^{m} P_i = 1 \) and Corollary 5 implies that \( G \) is a sub-direct product of a finite number of prime groups with \( \text{Min} - \sigma^n \) (respectively \( \text{Min} - sn \)). The converse is as in part(a).

We now come to our main theorem:

**Theorem 7** Let \( G \) be a group. Then

(a) \( G \in \text{Max} - \sigma^n(n \geq 3) \) (respectively \( \text{Max} - sn \)) if and only if

(i) \( \sigma(G) \) is soluble with \( \text{Max} \).

(ii) \( G/\sigma(G) \) is a sub-direct product of finitely many prime groups satisfying

\[ \text{Max} - \sigma^n(n \geq 3) \] (respectively \( \text{Max} - sn \))

(b) \( G \in \text{Min} - \sigma^n(n \geq 2) \) (respectively (or equivalently \( \text{Min} - sn \)) if and only if

(i) \( \sigma(G) \) is soluble with \( \text{Min} \).

(ii) \( G/\sigma(G) \) is a sub-direct product of finitely many prime groups satisfying

\[ \text{Min} - \sigma^n(n \geq 2) \] (respectively \( \text{Min} - sn \))

**Proof:** Combine Propositions 2 and 6.
Corollary 8: G is a finite group if and only if $\sigma(G)$ is finite and $G/\sigma(G)$ is a subdirect product of finitely many finite prime groups.

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