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On a Generalized Summation Formula

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Abstract: *In this paper the generalize a well-known asymptotic formula in two different ways.*

1. Introduction

Suppose $g(n)$ is an arithmetic function and $f(n)=\sum_{d|n} g(d)$.If the series $\sum_{n=1}^{\infty} g(n)$ converges absolutely it is well known that

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n) = \sum_{n=1}^{\infty}\frac{g(n)}{n} \quad (1.1)$$

ECKFORD COHEN [1] has generalized (1.1) in the form given below:

If the series $\sum_{n=1}^{\infty}\frac{g(n)}{n}$ is absolutely convergent and $g_s(n)=\sum_{d|n} g(d)\tau_s\left(\frac{n}{d}\right)$ where $\tau_s(n)$ (1.2)

Is defined by $\tau(n)=1, \tau_{s+1}(n)=\sum_{d|n}\tau_s(d)$, then

$$\lim_{x\to\infty} \frac{1}{x\log^{s-1}(x)}\sum_{n\le x} g_s(n) = \frac{1}{s-1}\sum_{n=1}^{\infty}\frac{g(n)}{n} \; (s=1,2,3,\ldots)$$

Later W.NARKIEWICZ [2] has given a simple proof of (1.2)

In the present paper we give generalizations of (1.1) in two different ways.

2. Main Results

In this section we prove the following.

Theorem Suppose g(n) is an arithmetic function and $f_k(n)=\sum_{d|n} g(d)\left(\frac{n}{d}\right)^k$ (2.1)

If $\sum_{n=1}^{\infty}\frac{g(n)}{n}$ is absolutely convergent then

$$\lim_{x\to\infty} \frac{1}{x^{k+1}}\sum_{n\le x} f_k(n) = \frac{1}{k+1}\sum_{n=1}^{\infty}\frac{g(n)}{n^{k+1}} \text{ for any } k\ge 0 \quad (2.2)$$

Proof: For any $k \geq 0$ note that

$$f_k(n) = \sum_{d\delta=n} g(d)\delta^k$$

$$= \sum_{d\le x} g(d)\left[\sum_{\delta\le\frac{x}{d}}\delta^k\right]$$

$$= \sum_{d \le x} g(d) \left(\frac{1}{k+1} (\frac{x}{d})^{k+1} + o(\frac{x}{d})^k \right)$$

$$= \frac{x}{k+1}^{k+1} \sum_{d \le x} \frac{g(d)}{d^{k+1}} + o(x^k \sum_{d \le x} \frac{|g(d)|}{d^k}),$$

which gives

$$\frac{1}{x^{k+1}} \sum_{n \le x} f_k(n) = \frac{1}{k+1} \sum_{n \le x} \frac{g(n)}{n^{k+1}} + O\left(\frac{1}{x} \sum_{n \le x} \left| \frac{g(n)}{n^k} \right| \right) \tag{2.3}$$

Now since $\sum_{n=1}^{\infty} \frac{g(n)}{n}$ is absolutely convergent, the series $\sum_{n=1}^{\infty} \frac{g(n)}{n}$ and $\sum_{n=1}^{\infty} \frac{|g(n)|}{n^{k+1}}$ are both convergent. Therefore taking limits as $x \to \infty$ in (2.3) we get

$$\lim_{x \to \infty} \frac{1}{x^{k+1}} \sum_{n \le x} f_k(n) = \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{g(n)}{n^{k+1}},$$

proving the theorem.

Remark: In the case k = 0 Theorem 2.1 gives (1.1). (2.4)

We can further generalize Theorem 2.1 in a different way as follows.

Theorem: Suppose that h (n) is an arithmetic function such that (2.5)

$$\sum_{n \le x} h(n) = A.x^s + B(x) \text{ where } B(x) = o(x^s) \tag{2.6}$$

Let g(n) be an arithmetic function such that

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} \textit{ converges absolutely.} \tag{2.7}$$

If f(n)= (g*h) (n) then

$$\lim_{x \to \infty} \frac{1}{x^s} \sum_{n \le x} f(n) = A \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$
(2.8)

Proof: Since $f(n) = \sum_{d\delta = n} g(d) h(\delta)$ we get by (2.6),

$$\sum_{n \le x} f(n) = \sum_{d\delta \le x} g(d) h(\delta)$$

$$= \sum_{d \le x} g(d) \left(\sum_{\delta \le \frac{x}{d}} h(\delta) \right)$$

$$= \sum_{d \le x} g(d) \left[A(\frac{x}{d})^s + B\,(\frac{x}{d}) \right]$$

$$= \text{A } x^s \sum_{d \le x} \frac{g(d)}{d^s} + \sum_{d \le x} B(\frac{x}{d}) g(d),$$

which gives

$$\frac{1}{x^s} \sum_{n \le x} f(n) = A \sum_{n \le x} \frac{g(n)}{n^s} + \frac{1}{x^s} \sum_{n \le x} B\left(\frac{x}{n}\right) g(n) \tag{2.9}$$

$$= A \sum_{n \le x} \frac{g(n)}{n^s} + o\left(\sum_{n \le x} \frac{g(n)}{n^s} \right)$$

since $\frac{B(x)}{x^s} \to 0$ as $x \to \infty$. Now taking limits as $x \to \infty$ in (2.9), we get Theorem 2.8.

Remark: Taking $h(n) = 1 \text{ for all } n$ in (2.8) we get (1.1). Also the case h(n)=n^k gives the Theorem 2.1. (2.10)

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