On Functions $A_f^S$, $G_f^S$ and $H_f^S$

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Abstract: In this paper we introduce the functions $A_f^S$, $G_f^S$ and $H_f^S$ for any multiplicative function $f$ and for any regular convolution $S$ and obtain a relation between them.

1. INTRODUCTION

For any arithmetic function $f$ let functions $A_f$, $G_f$ and $H_f$ be

1.1 \[ A_f(n) = \frac{1}{\tau(n)} \sum_{d \mid n} f(d) \]

1.2 \[ G_f(n) = \left[ \prod_{d \mid n} f(d) \right]^{\frac{1}{\tau(n)}} \]

and

1.3 \[ \frac{1}{H_f(n)} = \frac{1}{\tau(n)} \sum_{d \mid n} \frac{1}{f(d)}, \]

where $\tau(n)$ is the number of positive divisors of $n$.

Note that the function $A_f(n)$, $G_f(n)$ and $H_f(n)$ are respectively the arithmetic mean, geometric mean and harmonic mean of the function values of $f$ at various positive divisors of $n$.

In 1974 A. C. Vasu [2] has considered these functions and proved the following

1.4 If $f$ is multiplicative so are $A_f(n)$, $G_f(n)$ and $H_f(n)$

1.5 If $f$ is completely multiplicative then

\[ G_f(n) = A_f(n) \cdot H_f(n) = f(n) \]

Let $F$ be the set of all arithmetic functions

In this paper we introduce functions $A_f^S$, $G_f^S$ and $H_f^S$ for any $f \in F$ corresponding to regular divisors of $S_n$ of $n$ (defined below) and establish results which generalize (1.4) and (1.5)

2. PRELIMINARIES

Let $F$ be the class of all arithmetic functions. For any positive integer $n$ let $S_n$ denote a set of positive divisors of $n$. For $f, g \in F$ their $S$-product or $S$-convolution $f \bar{S} g$, is defined by

\[ (f \bar{S} g)(n) = \sum_{d \in S_n} f(d)g \left( \frac{n}{d} \right) \]
where the sum is over the divisors \( d \in S_n \).

2.1 The \( S \)-product is said to be regular if it satisfies the following conditions

(i) \( \langle F, +, S \rangle \) is a commutative ring with unity

(ii) \( f \bar{S} g \) is multiplicative whenever \( f \) and \( g \) are.

(iii) The arithmetic function \( u(n) = 1 \) for all \( n \) has inverse \( \mu_s \in F \) relative to \( \bar{S} \) (that is, \( u \bar{S} \mu_s = e \)) and \( \mu_s(n) = 0 \) or \(-1\) when \( n \) is a prime power. \( \mu_s \) is called the \( S \)-analogue of the Mobius function \( \mu \).

NARKIEWICZ [1] has characterized regular convolutions as follows:

2.2 Theorem. A \( S \)-convolution is regular if and only if the sets \( S_n \) have the following properties:

(i) \( d \in S_m, m \in S_n \iff d \in S_n, \frac{m}{d} \in S_n \)

(ii) \( d \in S_n \Rightarrow \frac{n}{d} \in S_n \)

(iii) \{1, n\} \subseteq S_n \) for every \( n \)

(iv) \( S_{mn} = S_m S_n = \{ab : a \in S_m, b \in S_n\} \) whenever \( \gcd(m,n) = 1 \)

(v) For every prime power \( p^\alpha \) we have
\[
S_{p^\alpha} = \{1, p, p^2, \ldots, p^\alpha\} \quad \text{or} \quad \alpha \text{ for some positive integer } t \quad \text{and} \quad p' \in S_{p^t}, p^{2t} \in S_{p^{2t}}, \ldots.
\]

2.3 Definition. If \( \bar{S} \) is a regular convolution the elements of \( S_n \) will be called regular divisors of \( n \). The number of \( S \)-divisors of \( n \) is denoted by \( \tau_S(n) \).

Since the Dirichlet convolution and the unitary convolution are both regular, the elements of \( D_n \) (the set of all positive divisors of \( n \)) and \( U_n \) (the set of all unitary divisors of \( n \)) are regular divisors of \( n \).

2.4: For any prime power \( p^\alpha \), the least positive integer \( t \) such that \( p' \in S_{p^\alpha} \) is called the type of \( p^\alpha \) relative to \( S \) and is denoted by \( t_S(p^\alpha) \).

If \( t = t_S(p^\alpha) \) it follows from the Theorem 2.2 (v) that \( t \mid a \) whenever \( p^\alpha \in S_{p^t} \). Clearly we have

\[
(2.5) \quad \tau_S(n) = \sum_{d \mid n} 1
\]

\[
(2.6) \quad \tau_S = u \bar{S} u \quad \text{where} \quad u \text{ as is in Definition 2.1 (iii) and since} \quad u \text{ is multiplicative, it follows from} \quad 2.1 (ii) \quad \text{that} \quad \tau_S \text{ is multiplicative.}
\]

Also

\[
(2.7) \quad \tau_S(p^\alpha) = \frac{\alpha}{t_S(p^\alpha)} + 1 \text{ for any prime power } p^\alpha.
\]

3. Main Results

Suppose \( S_n \) is a set of regular divisors of \( n \). For any arithmetic function \( f \) we define

\( A_f^S \), \( G_f^S \) and \( H_f^S \) by
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\begin{equation}
A_f^S(n) = \frac{1}{\tau_S(n)} \sum_{d \in S_n} f(d)
\end{equation}

\begin{equation}
G_f^S(n) = \left[ \prod_{d \in S_n} f(d) \right]^{1/\tau_S(n)}
\end{equation}

And whenever $f$ is nowhere zero

\begin{equation}
\frac{1}{H_f^S(n)} = \frac{1}{\tau_S(n)} \sum_{d \in S_n} \frac{1}{f(d)}
\end{equation}

Note that $A_f^S(n) = A_f(n)$, $G_f^S(n) = G_f(n)$ and $H_f^S(n) = H_f(n)$ where $D_n$ denotes the set of all positive divisors of $n$.

### 3.4 Theorem

If $f$ is multiplicative then $A_f^S$, $G_f^S$ and $H_f^S$ are all multiplicative.

**Proof:** By (2.6), $\tau_S$ is multiplicative. Again if $f$ is multiplicative it follows by (2.1)(ii), that $fS\bar{u}$ is also multiplicative. Therefore $A_f^S(n) = \frac{1}{\tau_S(n)} (fS\bar{u})(n)$ is also multiplicative. To prove the multiplicativity of $G_f^S$, note that for any $n$ with $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r}$, we have

\begin{equation}
G_f^S(n) = \left[ \prod_{d \in S_n} f(d) \right]^{1/\tau_S(n)}
\end{equation}

Now using 2.2 (iii) each $d \in S_{p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r}}$ can be written uniquely as $d = d_1d_2\ldots d_r$ where $d_i \in S_{p_i^{e_i}}$ for $i = 1, 2, 3, \ldots, r$ and $(d_i, d_j) = 1$ for $i \neq j$. Therefore (3.5) can be written as

\begin{equation}
G_f^S(n) = \prod_{i=1}^{r} \prod_{d_i \in S_{p_i^{e_i}}} f(d_i) = \prod_{i=1}^{r} G_f^S(p_i^{e_i})
\end{equation}

Thus $G_f^S(n)$ is multiplicative.

Observe that $H_f^S(n)$ can be defined only for the function $f$ which are nowhere zero. If $f$ is nowhere zero and multiplicative then so is $1/f$. Hence $A_f^S = H_f^S$.

Therefore $H_f^S$ is multiplicative.

### 3.6 Theorem

If $f$ is completely multiplicative then

\begin{equation}
\left[ G_f^S(n) \right]^2 = A_f^S(n) \cdot H_f^S(n) = f(n) \text{ for all } n.
\end{equation}

**Proof:** By Theorem 3.4 either side of (3.7) is multiplicative and therefore it is enough to verify the identity (3.7) in the case $n = p^\alpha$.

By (3.3) and the complete multiplicativity of $f(n)$ and since a $S$-convolution is regular if and
only if the set $S_n$ have the following the property, $d \in S_n \Rightarrow \frac{n}{d} \in S_n$

We have

$$\frac{1}{H_j^S(p^\alpha)} = \frac{1}{\tau_j(p^\alpha)} \sum_{d \in S^S_{\alpha}} f(d)$$

$$= \frac{1}{\tau_j(p^\alpha)} \sum_{d \in S^S_{\alpha}} \frac{f(p^\alpha)}{d}$$

$$= \frac{1}{f(p^\alpha)} \tau_j(p^\alpha) \sum_{d \in S^S_{\alpha}} \frac{p^\alpha}{d}$$

$$= \frac{1}{f(p^\alpha)} \tau_j(p^\alpha) \sum_{d \in S^S_{\alpha}} f(d)$$

$$= \frac{1}{f(p^\alpha)} A_j^S(p^\alpha)$$

Thus

(3.8) \quad A_j^S(p^\alpha) \cdot H_j^S(p^\alpha) = f(p^\alpha)

which gives the second part of the identity (3.7)

Again

$$G_j^S(p^\alpha) = \prod_{d \in S^S_{\alpha}} \left[ \frac{1}{\tau_j(p^\alpha)} \right]$$

$$= \prod_{0 \leq \beta \leq \alpha_{\beta \alpha}} \left[ \frac{1}{\tau_j(p^\alpha)} \right]$$

which gives

(3.9) \quad G_j^S(p^\alpha) = \left\{ f(p) \left[ \sum_{0 \leq \beta \leq \alpha_{\beta \alpha}} \beta \right] \right\}^{\frac{1}{\tau_j(p^\alpha)}}

But we have $S_{\alpha_{\beta \alpha}} = \{1, p^\beta, p^{2\beta}, \ldots, p^{\alpha_{\beta \alpha}}\}$ where $rt = \alpha$ and $t = t_j(p^\alpha)$

Therefore by (3.4) we get

(3.10) \quad \sum_{0 \leq \beta \leq \alpha_{\beta \alpha}} \beta = 0 + t + 2t + \ldots + rt

$$= r \left( \frac{r + 1}{2} \right) t$$

$$= \frac{\alpha}{2} \left( \frac{\alpha}{t} + 1 \right)$$
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\[ = \frac{\alpha}{2} \tau_S(p^\alpha) \]

From (3.9) and (3.10) we get

\[ G^S_f(p^\alpha) = \left[ f(p)^{\frac{\sigma}{\tau_S(p^\alpha)}} \right]^{\frac{1}{\tau_S(p^\alpha)}} \]

\[ = \left[ f(p) \right]^\frac{\sigma}{\tau} \]

which gives (3.7)

3.11 Remark: In the case $S_a = D_a$, Theorem 3.4 and Theorem 3.6 respectively give (1.4) and (1.5)

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REFERENCES