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Operatorial Chebyshev Spectral Method with Algebraic Singularities for Abel Integral Equations

Damian Trif

Department of Mathematics
Babes-Bolyai University
Cluj-Napoca,Romania
dtrif@math.ubbcluj.ro, dvtrif@gmail.com

Abstract: *The paper extends the applicability of our Matlab package Chebpack to find (generalized) Chebyshev polynomial approximations of the solutions of some Abel integral equations with algebraic singularities. Numerical examples confirm the efficiency of this approach.*

Keywords: *Abel integral equations, Chebyshev polynomials, algebraic singularities, spectral methods, Matlab.*

1. Introduction

There is a comprehensive literature on the theory, applications and numerical methods of the fractional calculus. For a brief history we refer to our open access paper Trif [1] and references therein. We must remark that any algorithm using a discretization of a non-integer derivative has to take into account its non-local structure which imply, in general, high storage requirements and a great overall complexity of the algorithm.

Our Matlab package *Chebpack*, see Trif [2], is based on the operational form of the Chebyshev spectral tau method and its main advantage is a unified approach for initial value problems, boundary value problems, eigenproblems, nonlocal problems for ordinary, fractional or distributed order differential equations. *Chebpack* assumes the representation of the unknown functions in truncated Chebyshev polynomials series

$$y(x) \approx y_{n-1}(x) = \frac{1}{2}c_0 T_0(x) + c_1 T_1(x) + \cdots + c_{n-1} T_{n-1}(x), \quad x \in [-1,1]. \tag{1}$$

If $y(x)$ is a smooth function then the above approximation is *spectrally accurate*, i.e. the approximation error decreases faster than any power of $1/n$ when $n \to \infty$. If $y(x)$ is not smooth, as it often happens with the solutions of fractional differential equations, the approximation may require large values of n as it can be seen in some examples in Trif [1].

The aim of the presented paper is to extend the capabilities of *Chebpack* to spectrally approximate functions with algebraic singularities such that $y(x) = x^q z(x)$, $q \geq 0$ where $z(x)$ is a well-behaved function. The idea is to use the above Chebyshev spectral approximation only for the function z .

The paper is structured as follows: in section 1 we describe the operational Chebyshev spectral method, in section 2 we give the discretization of the fractional integral operators by using the above idea and in section 3 we give numerical examples to illustrate the facilities of *Chebpack* for fractional calculus of functions with algebraic singularities. All the necessary Matlab code for reproducing the examples are now part of an updated version of *Chebpack* [2], in the folder Examples, subfolder Fractional differential equations.

2. The Operatorial Chebyshev Spectral Method

A sufficiently well-behaved function $y(x)$ can be accurately approximated by its physical representation $y(x_1), y(x_2), \ldots, y(x_n)$ of values of y at the given gridpoints $\bar{x} = (x_1, \ldots, x_n)$, for example at the Chebyshev points of the first or second kind

$$x_k^{(1)} = -\cos\frac{(2k-1)\pi}{2n},\ x_k^{(2)} = -\cos\frac{(k-1)\pi}{n-1},\ k = 1, \ldots, n. \tag{2}$$

The function $y(x)$ can also be accurately approximated by its spectral representation $\bar{c} = \{c_0, c_1, \ldots, c_{n-1}\}$ where $y_{n-1}(x)$ given by (1) is the unique polynomial obtained by interpolating $y(x)$ through the points $\bar{x}$. Of course, the Chebyshev polynomials are defined on $dom = [-1,1]$ but any interval $dom = [a,b]$ can be shifted to $[-1,1]$ and we may use the shifted Chebyshev polynomials

$$T_k^*(x) = T_k(\alpha x + \beta),\ \alpha = \frac{2}{b-a},\ \beta = -\frac{b+a}{b-a},\ k = 0,1,\ldots,n-1. \tag{3}$$

The code `[x,w]=pd(n,dom,kind)` of *Chebpack* calculates the n Chebyshev points $\bar{x}$ of the corresponding *kind* on *dom* as a column vector and the *n* quadrature weights $\bar{w}$ as a row vector for the Clenshaw-Curtis quadrature formula

$$\int_a^b y(x)dx \approx \sum_{j=1}^{n} w_j y(x_j). \tag{4}$$

The fast conversion between the spectral representation $\bar{c}$ of a function *y* and its physical values $\bar{v} = y(\bar{x})$ is performed by the functions `v=t2x(c,kind)` and `c=x2t(v,kind)` based on the Fast Chebyshev Transform.

If a function *y* is given by its Chebyshev coefficients $\bar{c}$ and we need its values at some points $\bar{x}_c \in [a,b]$, a conversion matrix T_c is obtained from the code `Tc=cpv(n,xc,dom)`, where

$$T_c = \left[\frac{T_0}{2}, T_1(\bar{\xi}), \ldots, T_{n-1}(\bar{\xi})\right],\ \bar{\xi} = \frac{2\bar{x}_c}{b-a} - \frac{b+a}{b-a} \in [-1,1] \tag{5}$$

where $\bar{\xi}$ is transposed as a column vector and we have $y(\bar{x}_c) = T_c \cdot \bar{c}$. If `T=cpv(n,x,dom)` where $\bar{x}$ are the Chebyshev points, then the matrix *T* performs the (non fast, but useful if $n \le 256$) conversion between the coefficients $\bar{c}$ of the function *y* and the values $\bar{v} = y(\bar{x})$ through the formulas $\bar{v} = T \cdot \bar{c}$ and $\bar{c} = T^{-1} \cdot \bar{v}$.

The differentiation is discretized by a differentiation matrix *D* given by the code `D=deriv(n,dom)`. If $\bar{c}$ is the column of Chebyshev coefficients of a function $y(x)$, then $D \cdot \bar{c}$ is the column of the Chebyshev coefficients of the derivative $\frac{dy}{dx}$. The definition of *D* is based on the recurrence relations

$$T_0 = T_1',\ T_1 = \frac{T_2'}{4}, T_k = \frac{T_{k+1}'}{2(k+1)} - \frac{T_{k-1}'}{2(k-1)},\ k = 2,3,\ldots,\ x \in [-1,1]. \tag{6}$$

Similarly, the code `[J,J0]=prim(n,dom)` calculates the integration matrix *J* such that the coefficients of a primitive of $y(x)$ are $J \cdot \bar{c}$. The coefficients of the particular primitive vanishing at $a = dom(1)$ are obtained by using $J_0 \cdot \overline{c}$. Another useful code is `X=mult(n,dom).` Then $X \cdot \overline{c}$ is the column of the Chebyshev coefficients of the multiplication by the independent variable $x \cdot y(x)$ for $x \in dom$.

If $L : C^{\infty}(-1,1) \to C^{\infty}(-1,1)$ is a linear operator then let $\overline{c}, \overline{u}$ be the corresponding coefficients of $y(x)$ and $L(y(x))$. The matrix $\overline{L}$ that maps $\overline{c}$ into $\overline{u} = \overline{L} \cdot \overline{c}$ is the Chebyshev approximation of L. *Chebpack* implements the Chebyshev spectral method as a *Lanczos' tau method* where we work in the spectral space of the coefficients. The linear operators of the differential or integral problem, such as differentiation, integration, product with the independent variable or modified argument, are discretized to their corresponding approximating matrices. The final form of the linear problem $L(y)(x) = f(x)$ becomes, after the discretization, a pure algebraic linear problem in an operatorial form $\overline{L} \cdot \overline{c} = \overline{f}$ with the supplementary conditions of the continuous problem included. It is important to remark that linear operators are better represented in the spectral space of the coefficients, while the nonlinear operators are easily handled in the physical space of the values. All the above codes take into account a general *dom*, see the open access chapter Trif [3] for more details. The basic results for the convergence of the above spectral approximations are given by the Theorems 8.1, 8.2, and 21.1 from Trefethen [4].

3. THE FRACTIONAL OPERATORS

The Riemann--Liouville fractional integral operator of order q is defined by

$$J^q y(x) = \frac{1}{\Gamma(q)} \int_0^x \frac{y(t)dt}{(x-t)^{1-q}},\ 0 < q < 1,\ 0 \le x \le b. \tag{7}$$

If $y \in L^1[0,b]$ has the spectral approximation

$$y(x) \approx \sum_{k=0}^{n-1}{}' c_k T_k\left(\frac{2}{b}x - 1\right), \tag{8}$$

where the prime sign denotes the summation whose first term is halved, then

$$J^q y(x_j) \approx \frac{1}{\Gamma(q)} \sum_{k=0}^{n-1}{}' c_k \int_0^{x_j} \frac{T_k\left(\frac{2}{b}t - 1\right)}{(x_j - t)^{1-q}} dt,\ j = 1, \ldots, n \tag{9}$$

approximates the physical values $\overline{v}$ of $J^q y(\overline{x})$ at the Chebyshev points $\overline{x} = (x_j)_{j=1,\ldots,n}$. Consequently, the spectral approximation of $J^q y(x)$ is given by the Chebyshev coefficients $T^{-1}\overline{v} \equiv I \cdot \overline{c}$, where T is given by (5), $\overline{c}$ is the column vector of the coefficients of $y(x)$ and I is the basic integration matrix calculated in [1] by the code `I=fracbas(x,dom,q)` for $0 < q < 1$.

Many examples in [1] confirm the efficiency of this approach. But in [1] there are also examples where this method is not suitable. Such an example is the Abel integral equation of the second kind

$$y(x) + \int_0^x \frac{y(t)dt}{\sqrt{x-t}} = 2\sqrt{x} \tag{10}$$

with the exact solution $y_{ex}(x) = 1 - e^{\pi x} erfc(\sqrt{\pi x})$. The above standard method gives for $n = 64$ the numerical values of the solution with an error of 1.7543×10^{-4} and needs a computing time of 6×10^{-3} seconds. For $n = 512$ the error becomes 2.7658×10^{-6} for an elapsed time of 3.3

seconds. In this case the solution $y_{ex}(x)$ has singularities of lower-order derivatives and a good approximation of it requires an excessively large value of n. Obviously, in practical applications it is however required to approximate the fractional integrals or derivatives of such badly-behaved functions.

If we have to solve the Abel integral equation of the first kind

$$\int_0^x \frac{y(t)dt}{(x-t)^q} = f(x),\ 0<q<1,\ x\in[0,1] \tag{11}$$

then let us suppose, cf. [5], that $f(x)$ can be approximated accurately by

$$f(x) \approx x^{\beta}\sum_{k=0}^{n-1}{}' c_k T_k(1-2x),\ \beta>0. \tag{12}$$

The solution of

$$\int_0^x \frac{g_k(t)dt}{(x-t)^q} = x^{\beta}T_k(1-2x),\ k=0,\ldots,n-1 \tag{13}$$

Is given by

$$g_k(x) = \frac{x^{q+\beta-1}}{\Gamma(q+\beta)}\frac{\Gamma(1+\beta)}{\Gamma(1-q)}\ {}_3F_2\left(\begin{matrix}-k,k,\beta+1 \\ \frac{1}{2},q+\beta\end{matrix}\ ;\ x\right) \tag{14}$$

and, consequently,

$$y(x) = \frac{x^{q+\beta-1}}{\Gamma(q+\beta)}\frac{\Gamma(1+\beta)}{\Gamma(1-q)}\sum_{k=0}^{n-1}{}' c_k \cdot {}_3F_2\left(\begin{matrix}-k,k,\beta+1 \\ \frac{1}{2},q+\beta\end{matrix}\ ;\ x\right). \tag{15}$$

The above hypergeometric functions ${}_3F_2$ can be calculated directly by Matlab or by a recurrence formula from [5]. Obviously, if $\overline{c}$ is the column vector of the Chebyshev coefficients from (12) then (15) is of the form $y(\overline{x}) = I\cdot\overline{c}$, where *I* is the solution matrix of dimension *n* from (14) and $\overline{x}$ is the column of the Chebyshev nodes. Note that in order to eliminate the strong singularities at the origin of this form of the solution, we must choose β such that $q+\beta-1\geq 0$.

A test problem is the Abel integral equation of the first kind (see [5])

$$\int_0^x \frac{y(t)dt}{\sqrt{x-t}} = e^x-1,\quad x\in[0,1] \tag{16}$$

with the exact solution $y_{ex}(x) = \frac{e^x}{\sqrt{\pi}}erf\left(\sqrt{x}\right)$. We put the r.h.s under the form

$$e^x-1 = x\cdot\frac{e^x-1}{x} = x\cdot g(x) \tag{17}$$

so that $q=\frac{1}{2}, \beta=1$ and then $q+\beta-1\geq 0$ and we use the Chebyshev spectral approximation for the function $g(x)$. For $n=12$ the error is about 2.66×10^{-15} and the computing time is 0.5 seconds. The *Chebpack* code is `test_as1.m` (where, for simplicity, the hypergeometric functions are calculated directly by Matlab).

Let us return to the Abel integral equation of the second kind (10). *Chebpack* can handle this kind of problems by using the formulas (13) and (14) under the form

$$\int_0^x \frac{t^\gamma f_{j,\gamma}(t)}{(x-t)^{1-q}}dt = x^{\gamma+q}T_j(1-2x),\ j=0,\dots,n-1,\ x\in(0,1),\ q\in(0,1),\ \gamma>0, \tag{18}$$

where

$$f_{j,\gamma}(t)=\frac{{}_3F_2\left(\begin{matrix}-j, j,\gamma+q+1 \\ \frac{1}{2},\gamma+1\end{matrix};\ t\right)}{B(q,\gamma+1)},\ j=0,\dots,n-1. \tag{19}$$

are in fact polynomials.

If we consider the equation

$$y(x)+\lambda\int_0^x \frac{y(t)}{(x-t)^{1-q}}dt = f(x),\ \ x\in[0,1] \tag{20}$$

of the type $\left(I+\lambda\Gamma(q)J^q\right)y(x)=f(x)$, the formal solution is (see Gorenflo & Mainardi [6])

$$y(x)=\left(I+\lambda\Gamma(q)J^q\right)^{-1}f(x)=\left(I+\sum_{k=1}^{\infty}\left(-\lambda\Gamma(q)\right)^k J^{kq}\right)f(x). \tag{21}$$

An iterative form of (21) is

$$\begin{aligned} &y_0(x)=f(x),\\ &y_k(x)=-\lambda\Gamma(q)J^q y_{k-1}(x),\ k=1,2,\dots,N,\\ &y(x)=\sum_{k=0}^{N} y_k(x). \end{aligned} \tag{22}$$

If we consider functions f such that

$$f(x)=x^\gamma {\sum_{j=0}^{n-1}}' c_j T_j(1-2x)=x^\gamma\sum_{j=0}^{n-1}a_j f_{j,\gamma}(x),\ \gamma\geq 0, \tag{23}$$

the code `TH=convert(q,gam,x)` implements the recurrence formula from [5] for the fast computation of the hypergeometric functions in (19).

This code gives the conversion matrix TH between the two representations of $f(x)$,

$$f(\bar{x})=TT\cdot\bar{c}=TH\cdot\bar{a} \tag{24}$$

where $\bar{c}$ and $\bar{a}$ are the column vectors of the Chebyshev and the hypergeometric coefficients of f, respectively, and TT is derived from the matrix T by changing the sign of the odd lines. In each iteration (22), y_{k-1} is converted from the Chebyshev to hypergeometric form, by applying the corresponding formula (18) for each j - term and obtaining y_k in Chebyshev series form, but now with the factor $x^{\gamma+q}$ included. Finally, we sum the physical values of the intermediate terms y_k until $\|y_k\| < \varepsilon$ for some norm.

The above algorithm involves the inversion of the matrix TH and the quality of the inversion depends on the condition number of TH. Table 1 shows the condition number of this matrix for $q = \gamma = 0.5$ compared to that of T and Vandermonde matrix V (used in calculating interpolating polynomial coefficients directly at Chebyshev interpolation nodes).

Table 1. *Condition numbers for the matrices T, TH, V*

	T	TH ($q = \gamma = 0.5$)	V
$n = 16$	2.1662	20.3623	1.16×10^{11}
$n = 32$	2.1195	42.5702	7.55×10^{18}
$n = 64$	2.0855	87.3065	3.83×10^{19}

We remark that the same Chebyshev spectral method, but not in operational form and without applications to Abel integral equations is used in Theorem 3.2 from Sugiura & Hasegawa [7] where the uniform approximation of the fractional derivatives is also proved.

The code `test_as2.m` applies this procedure for $\gamma = 0.5$ to solve the equation (10) and gives a numerical solution with an error of 3×10^{-15} for $n = 16$ after 54 iterations and 0.01 seconds elapsed time. This shows a strong improvement in efficiency compared to the standard method of *Chebpack.*

4. Numerical Examples

This section contains some examples for fractional integral equations with algebraic singularities.

Example 1. Consider the first kind Abel integral equation from [8]

$$\int_0^x \frac{y(t)}{(x-t)^{\frac{1}{3}}} dt = x^{\frac{7}{6}}, \quad x \in [0,1]. \tag{25}$$

For $n = 16$ and $\beta = \frac{7}{6}$ the code `example_as1.m` (similar to `test_as1.m` with the new data) gives the exact solution $y_{ex}(x) = C\sqrt{x}$ where $C = 7\Gamma(1/6)/\left(18\sqrt{\pi}\Gamma(2/3)\right)$.

Example 2. Consider the second kind Abel integral equation from [9]

$$y(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = e^x\left(1 + \sqrt{\pi} erf\left(\sqrt{x}\right)\right), \tag{26}$$

with the exact solution $y_{ex}(x) = e^x$. The code `example_as2.m` applies twice the iterative method separately for the right-hand side $f_1(x) = e^x$ (a well behaved function, $\gamma = 0$) and for $f_2(x) = \sqrt{\pi} e^x erf\left(\sqrt{x}\right)$, which is of the form $\sqrt{x}$ times a well behaved function ($\gamma = 0.5$). The superposition of the corresponding solutions $y_1(x) + y_2(x)$ for $n = 8$ approximates the exact solution with an error of 6.5×10^{-10} in 0.011 seconds elapsed time. For $n = 16$ the error becomes 4×10^{-15} and the elapsed time becomes 0.016 seconds. Both cases needed 55 iterations in (22).

Example 3. Consider the fractional differential problem from [10]

$$ {}^{C}D^{q} y + y = 0,\ y(0) = 1,\ q = \frac{1}{2} \tag{27} $$

with the exact solution

$$ y(x) = E_{\frac{1}{2}}\left(-\sqrt{x}\right) = 1 - \frac{\sqrt{x}}{\Gamma\left(\frac{3}{2}\right)} + \frac{x}{\Gamma(2)} - \frac{x\sqrt{x}}{\Gamma\left(\frac{5}{2}\right)} + \cdots \tag{28} $$

The problem can be transformed to $y(x) + J^{\frac{1}{2}} y(x) = 1$.

The standard procedure of *Chebpack* (code `example_as3.m`) gives for $n = 32$ an error of 2.3×10^{-4} after an elapsed time of 0.002 seconds. For $n = 512$ the error becomes 9×10^{-7} after an elapsed time of 3.15 seconds.

The improved procedure (taking into account the algebraic singularity of the solution at the origin) `example_as3_improved.m` gives for $n = 16$ an error of 6×10^{-16} after 36 iterations and an elapsed time of 0.014 seconds.

Example 4. Consider the nonlinear Abel type integral equation from [11], [12] obtained by transforming a Lighthill's problem (1950) which describes the temperature distribution of the surface of a projectile moving through a laminar layer

$$ y(x) = 1 - \frac{\sqrt{3}}{\pi} \int_0^x \frac{t^{\frac{1}{3}} y(t)^4}{(x-t)^{\frac{2}{3}}} dt,\quad x \in [0,1]. \tag{29} $$

According to the formula (18), we look for the solution as

$$ y(x) = p(x) + x^{\frac{1}{3}} q(x) + x^{\frac{2}{3}} r(x). \tag{30} $$

where $p(x), q(x)$ and $r(x)$ are polynomials. A small piece of program `nonlin.m` calculates the physical values of the polynomials $P(x), Q(x)$ and $R(x)$ such that

$$ y(x)^4 = P(x) + x^{\frac{1}{3}} Q(x) + x^{\frac{2}{3}} R(x). \tag{31} $$

Next, by using the conversion matrices `TH=convert(q,gam,x)` P, Q and R can be written as a combination of hypergeometric functions. The integral in (29) becomes

$$ f(\bar{x}) = \int_0^{x} \frac{t^{\frac{1}{3}} \sum_{j=0}^{n-1} a_j^{(0)} f_{j,\frac{1}{3}}^{(0)}(t) + t^{\frac{2}{3}} \sum_{j=0}^{n-1} a_j^{(1)} f_{j,\frac{2}{3}}^{(1)}(t) + t \sum_{j=0}^{n-1} a_j^{(2)} f_{j,1}^{(2)}(t)}{(\bar{x}-t)^{1-\frac{1}{3}}} dt = $$

$$ = (\bar{x})^{\frac{2}{3}} \cdot \sum_{j=0}^{n-1} a_j^{(0)} T_j(1-2\bar{x}) + \bar{x} \cdot \sum_{j=0}^{n-1} a_j^{(1)} T_j(1-2\bar{x}) + (\bar{x})^{\frac{4}{3}} \cdot \sum_{j=0}^{n-1} a_j^{(2)} T_j(1-2\bar{x}) \tag{32} $$

for $\bar{x} = [x_1, \ldots, x_n]$ and $f_{j,\gamma}(t)$ given by (19).

Finally, if we consider as unknowns the values of the polynomials $p(\bar{x}), q(\bar{x})$ and $r(\bar{x})$ concatenated in a long column vector $\bar{v}$ with *3n* entries, the above relations lead us to a nonlinear system $F(\bar{v}) = 0$ of the form

$$\bar{v} + \frac{\sqrt{3}}{\pi} f(\bar{x}) - \bar{1} = \bar{0},. \tag{33}$$

where $f(\bar{x})$ depends nonlinearly on $\bar{v}$. This system is then solved by `fsolve` of Matlab starting with an initial approximation of the solution as $y^{(0)}(\bar{x}) = \bar{1}$.

The code `example_as4.m` gives for *n=32* and an elapsed time 0.75 seconds the value $y(1) \approx 0.664857230096775$ while for *n=64* and the elapsed time 3.62 seconds gives the value $y(1) \approx 0.664857150875165$. The comparison values for this problem are the best one from [13], $y(1) \approx 0.6648571508$ and the best one from [11], $y(1) \approx 0.664859$. For *n=128* this *Chebpack* solution is graphically compared with that from [11] and with the asymptotic series approximation of Lighthill (see again [11]) in Fig. 1. For all these cases `fsolve` needs 6 iterations.

If we need an approximation on a larger interval $[0,b]$ with $b > 1$, then let $\bar{\xi} = \bar{x}/b$ and $\bar{\tau} = \bar{t}/b$ and (29) becomes

$$y(b\xi) = 1 - \frac{\sqrt{3}}{\pi} b^{\frac{2}{3}} \int_0^x \frac{\tau^{\frac{1}{3}} y(b\tau)^4}{(\xi - \tau)^{\frac{2}{3}}} d\tau. \tag{34}$$

With this small change, (33) with $\lambda = \frac{\sqrt{3}}{\pi} b^{\frac{2}{3}}$ instead of $\lambda = \frac{\sqrt{3}}{\pi}$ gives the numerical values $y(b\bar{\xi})$ of the solution on the larger interval $[0,b]$. The code `example_as4_long.m` performs these calculations for *n=128* and *b=25* in 8 iterations and the numerical solution on $[0,25]$ is in good concordance with the asymptotic series of Lighthill [11].

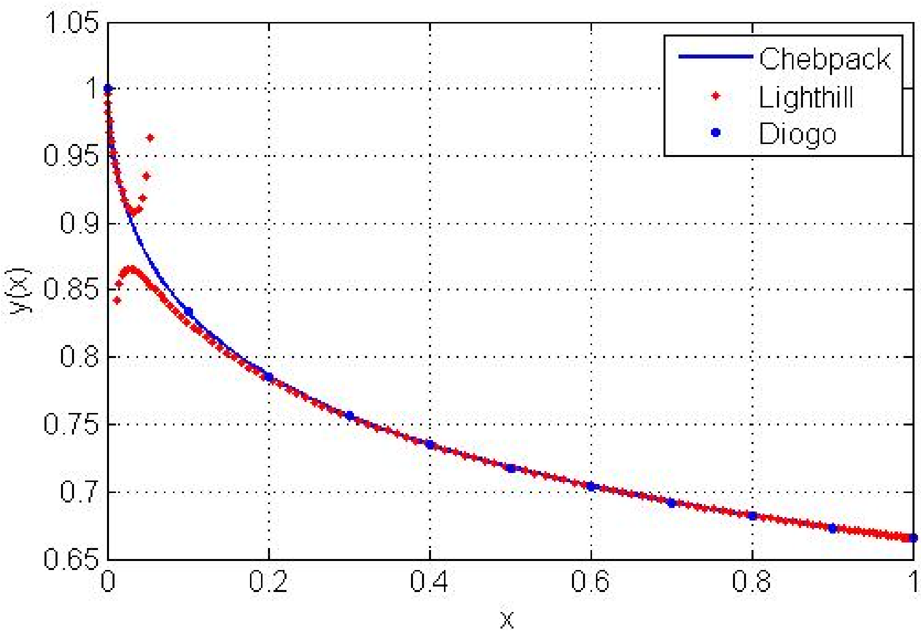

Figure 1. *Approximations for the Lighthill's equation*

5. Conclusion

The Chrbyshev spectral method implemented in our package *Chebpack* leads to very simple and efficient codes that can solve different kinds of problems for fractional differential equations in a unified approach. The Chebyshev grid points are automatically clustered near the left endpoint of the working interval but this is not enough for a good approximation if the solution has singularities at that point. The presented paper extends the capabilities of *Chebpack* to spectrally approximate such functions with algebraic singularities. All the necessary Matlab sources for reproducing the above tests and examples are now part of an updated version of *Chebpack* [2] in the folder Examples, subfolder Fractional differential equations.

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AUTHOR'S BIOGRAPHY

Senior Researcher (I)

Department of Mathematics,

Faculty of Mathematics and Computer Science,

Babes-Bolyai University of Cluj-Napoca, Romania