Laplacian Polynomial and Laplacian Energy of Some Cluster Graphs

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Abstract: The graphs with large number of edges are referred as graph representation of inorganic clusters, so called cluster graphs. H . B. Walikar and H .S. Ramane introduced class of graph obtained from complete graph by deleting edges. In this paper, the Laplacian polynomial and Laplacian energy of this class of graph is obtained.

Keywords: Laplacian polynomial and Laplacian energy of graph, cluster graphs.

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1. INTRODUCTION

The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories.

Let G be a simple graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) where \( n \) is the number of vertices of G. The adjacency matrix of a graph G is \( A(G) = [a_{ij}] \), where \( a_{ij} = 1 \) if \( v_i \) is adjacent to \( v_j \) and \( a_{ij} = 0 \), otherwise. The characteristic polynomial of a graph G is defined as

\[
\phi(G; \lambda) = \det(\lambda I - A(G))
\]

where I is the identity matrix of order \( n \).

The degree matrix of a graph G is the diagonal matrix \( D(G) = \text{diag}[d_i] \) where \( d_i = d_G(v_i) \). The matrix \( C(G) = D(G) - A(G) \) is called Laplacian matrix. It is also called as matrix of admittance due to its role in electrical theory [1]. The Laplacian polynomial of graph G is defined as

\[
C(G; \mu) = \det(\mu I - C(G))
\]

where I is the identity matrix of order \( n \). The roots \( \mu_1, \mu_2, \ldots, -\mu_n \) of \( C(G; \mu) \) are called the Laplacian eigenvalues of G, where \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \).

Laplacian energy is defined as

\[
CE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|
\]

Let \( K_n \) denote the complete graph on \( n \) vertices. The class of graphs defined [2] is as follows.

2. SOME CLUSTER GRAPHS

I.Gutman and L. Pavlovic [2] introduced four classes of graphs obtained from complete graph by deleting edges and analyzed their energies. For completeness we produce these here.

DEFINITION 1: Let \( v \) be a vertex of a complete graph \( K_n \), \( n \geq 3 \) and let \( e_i, i = 1, 2, \ldots, k \), \( 1 \leq k \leq n-1 \), be its distinct edges, all being incident to \( v \). The graph \( K a_n(k) \) is obtained by deleting \( e_i, i = 1, 2, \ldots, k \) from \( K_n \). In addition \( K a_n(0) \cong K_n \).

DEFINITION 2: Let \( f_i, i = 1, 2, \ldots, k \), \( 1 \leq k \leq \lfloor n/2 \rfloor \) be independent edges of the complete graph \( K_n \), \( n \geq 3 \). The graph \( K b_n(k) \) is obtained by deleting \( f_i, i = 1, 2, \ldots, k \) from \( K_n \). In addition \( K b_n(0) \cong K_n \).
DEFINITION 3: Let $V_k$ be a $k$-element subset of the vertex set of complete graph $K_n$, $2 \leq k \leq n$, $n \geq 3$. The graph $Kc_n(k)$ is obtained by deleting from $K_n$ all the edges connecting pairs of vertices from $V_k$. In addition $Kc_n(0) \equiv Kc_n(1) \equiv K_n$.

DEFINITION 4: Let $3 \leq k \leq n$, $n \geq 3$. The graph $Kd_n(k)$ is obtained from $K_n$, the edges belonging to a $k$-membered cycle.

H.S. Ramane and H.B. Walikar [3] has introduced another class of graph obtained from $K_n$ and is denoted by $Ka_n(p, k)$ which is as follows.

DEFINITION 5: Let $(K_p)_i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor n/p \rfloor$, $1 \leq p \leq n$, be independent complete graphs with $p$ vertices of the complete graph $K_n$, $n \geq 3$. The graph $Ka_n(p, k)$ is obtained from $K_n$ by deleting all edges of $(K_p)_i$, $i = 1, 2, \ldots, k$. In addition

$Ka_n(p, 0) \equiv Ka_n(0, k) \equiv Ka_n(0, 0) \equiv K_n$.

In this paper Laplacian polynomial and energy of $Ka_n(p, k)$ is obtained.

Note that the Laplacian polynomial and Laplacian energy of $Kb_n(k)$ and $Kc_n(k)$ [4] are particular cases of the graph $Ka_n(p, k)$.

**Theorem 1:** For $n \geq 3$, $1 \leq k \leq \lfloor n/p \rfloor$, $1 \leq p \leq n$.

$$C(Ka_n(p, k)) = \mu(\mu - n)^{(n-k)(p-1)-1} (\mu - n + p)^{k(p-1)}$$ \hspace{1cm} (1)

**Proof:** Without loss of generality we assume that the vertices of $(K_p)_i$ are $v_{m(i-1)+1}, v_{m(i-1)+2}, \ldots, v_{m(i-1)+m}$, $i = 1, 2, \ldots, k$.

In order to make the following result more compact, the auxiliary quantity $X$ is introduced

$X = \mu - n + p$.

Then the Laplacian polynomial of $Ka_n(p, k)$ is equal to the determinant

$$|X 0 0 \ldots 0 1 1 1 \ldots 1 1 1 1 \ldots 1 1 \ldots 1|
0 X 0 \ldots 0 1 1 1 \ldots 1 1 1 1 \ldots 1 1 \ldots 1
0 0 X \ldots 0 1 1 1 \ldots 1 1 1 1 \ldots 1 1 \ldots 1
\vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots
0 0 0 \ldots X 1 1 1 \ldots 1 1 1 1 \ldots 1 1 \ldots 1
1 1 1 \ldots 1 X 0 0 \ldots 0 1 1 1 \ldots 1 1 \ldots 1
1 1 1 \ldots 1 0 X 0 \ldots 0 1 1 1 \ldots 1 1 \ldots 1
1 1 1 \ldots 1 0 0 X \ldots 0 1 1 1 \ldots 1 1 \ldots 1
\vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots
1 1 1 \ldots 1 0 0 0 \ldots X 1 1 1 \ldots 1 1 \ldots 1
\vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots
1 1 1 \ldots 1 1 1 1 \ldots X 0 0 \ldots 0 1 \ldots 1
1 1 1 \ldots 1 1 1 1 \ldots 1 0 X 0 \ldots 0 1 \ldots 1
1 1 1 \ldots 1 1 1 1 \ldots 1 0 0 X \ldots 0 1 \ldots 1
\vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots
1 1 1 \ldots 1 1 1 1 \ldots 1 0 0 0 \ldots X 1 \ldots 1
1 1 1 \ldots 1 1 1 1 \ldots 1 0 0 0 \ldots X X - p + 1 \ldots 1
\vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots
1 1 1 \ldots 1 1 1 1 \ldots 1 1 1 1 \ldots 1 1 \ldots X - p + 1
\vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots
1 1 1 \ldots 1 1 1 1 \ldots 1 1 1 1 \ldots 1 1 \ldots X - p + 1$$

Subtract first column from 2, 3, \ldots, $n$ columns of (2) to obtain (3)
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\[
\begin{pmatrix}
X - X & \cdots & -X & 1 - X & 1 - X & \cdots & 1 - X & 1 - X \cdots & 1 - X & 1 - X \\
0 & X & \cdots & 0 & 1 & 1 \cdots & 1 & 1 \cdots & 1 & 1 \\
0 & 0 & \cdots & X & 1 & 1 \cdots & 1 & 1 \cdots & 1 & 1 \\
1 & 0 & \cdots & 0 & X - 1 & -1 \cdots & -1 & 0 \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & -1 & X - 1 \cdots & -1 & 0 \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & -1 & -1 \cdots & X - 1 & 0 \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 \cdots & 0 & 0 \cdots & 0 & X - p \\
1 & 0 & \cdots & 0 & 0 & 0 \cdots & 0 & 0 \cdots & 0 & X - p \\
\end{pmatrix} \quad (3)
\]

Add 2,3,…,n rows to first row of (3) to obtain (4)

\[
\begin{pmatrix}
X + n - p & 0 & \cdots & 0 & 0 & 0 \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & X & \cdots & 0 & 1 & 1 \cdots & 1 & 1 \cdots & 1 & 1 \\
0 & 0 & \cdots & X & 1 & 1 \cdots & 1 & 1 \cdots & 1 & 1 \\
1 & 0 & \cdots & 0 & X - 1 & -1 \cdots & -1 & 0 \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & -1 & X - 1 \cdots & -1 & 0 \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & -1 & -1 \cdots & X - 1 & 0 \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 \cdots & 0 & 0 \cdots & 0 & X - p \\
1 & 0 & \cdots & 0 & 0 & 0 \cdots & 0 & 0 \cdots & 0 & X - p \\
\end{pmatrix} \quad (4)
\]

Evidently, expression (4) is equal to (5)

\[
(X + n - p) (X - p)^{n-pk} X^{p-1} = \begin{pmatrix}
X - 1 & -1 & -1 \cdots & -1 \\
-1 & X - 1 & -1 \cdots & -1 \\
-1 & -1 & X - 1 \cdots & -1 \\
\cdots & \cdots & \cdots & \cdots \\
-1 & -1 & -1 \cdots & X - 1 \\
\end{pmatrix}^{k-1} \quad (5)
\]

Subtract first column from 2,3,…,p columns of (5) to obtain (6)

\[
(X + n - p) (X - p)^{n-pk} X^{p-1} = \begin{pmatrix}
X - 1 & -X & -X \cdots & -X \\
-1 & X & 0 \cdots & 0 \\
-1 & 0 & X \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-1 & 0 & 0 \cdots & X \\
\end{pmatrix}^{k-1} \quad (6)
\]

Add 2,3,…,p rows to first row of (6) to obtain (7)

\[
(X + n - p) (X - p)^{n-pk} X^{p-1} = \begin{pmatrix}
X - p & 0 & 0 \cdots & 0 \cdots & 0 \\
-1 & X & 0 \cdots & 0 \\
-1 & 0 & X \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-1 & 0 & 0 \cdots & X \\
\end{pmatrix}^{k-1} \quad (7)
\]

Expression (7) is equal to

\[
(X + n - p) (X - p)^{n-pk} X^{p-1} (X - p)^{k-1} (X^{p-1})^{k-1} \quad (8)
\]
On simplification expression (8) reduces to (9)

\[(X + n - p)(X - p)^{n-p+k+1}X^{k(p-1)} \quad (9)\]

This leads to the expression (1).

This completes the proof.

3. **Laplacian Spectra and Laplacian Energy of \(K_{\alpha_n}(p, k)\)**

**Corollary 2:**

For \(1 \leq k \leq \lfloor n/p \rfloor\), \(1 \leq p \leq n\), the spectrum of \(C(K_{\alpha_n}(p, k))\) consists of \(0, n \{\text{(n-k(p-1)-1) times}\} \) and \(n-p \{\text{k(p-1) times}\} \)

**Theorem 3:**

For \(1 \leq k \leq \lfloor n/p \rfloor\), \(1 \leq p \leq n\),

\[CE(K_{\alpha_n}(p, k)) = \frac{2}{n}[n(n-1) - pk(p-1) + k(n-pk)(p-1)^2] \quad (10)\]

**Proof:** The Laplacian energy is given by

\[CE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|\]

**Substituting the value of** \(2m\) **in the equation (11) we straightforwardly obtain (10).**

**Remarks:**

1. If \(k = 0\) equation (1) reduces to
   \[C(K_{\alpha_n}(p, 0)) = \mu(\mu - n)^{n-1} \text{ which is a Laplacian polynomial of } K_n.\]

2. If \(p = 1\), then equation (1) reduces to
   \[C(K_{\alpha_n}(1, k)) = \mu(\mu - n)^{n-1} \text{ which is a Laplacian polynomial of } K_n.\]

3. If \(p = n\) and \(k = 1\) then equation (1) reduces to
   \[C(K_{\alpha_n}(n, 1)) = \mu^n \text{ which is a Laplacian polynomial of } K_p, \text{ the complement of } K_p.\]

4. If \(p = 2\), \(k = n/2\) then equation (1) reduces to
   \[C(K_{\alpha_n}(2, n/2)) = \mu(\mu - n)^{n-1}(\mu - n + 2)^{n/2} \text{ which is a Laplacian polynomial of Cocktail party graph}\]

5. If \(n = pk\) then equation (1) reduces to
   \[C(K_{\alpha_n}(p, k)) = \mu(\mu - pk)^{k(p-1)}(\mu - p(k - 1))^{k(p-1)} \text{ which is a Laplacian polynomial of complete multipartite graph } K_{n_1, n_2, \ldots, n_k} \text{ where } n_1 = n_2 = \ldots = n_k = n/k\]

6. If \(p = 2\) and \(k = 1\) then equation (1) reduces to
   \[C(K_{\alpha_n}(2, 1)) = \mu(\mu - n)^{n-2}(\mu - n + 2) \text{ which is a Laplacian polynomial of } K_{\alpha_n}(1)\]

7. If \(p = 2\) then equation (1) reduces to
   \[C(K_{\alpha_n}(2, k)) = \mu(\mu - n)^{n-k+1}(\mu - n + 2)^k \text{ which is a Laplacian polynomial of } K_{\alpha_n}(k) \text{ [4].}\]

8. If \(k = 1\) then equation (1) reduces to
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\[ C(K_{n}(p, 1)) = \mu(\mu - n)^{n-p}(\mu - n + p)^{p-1} \] which is a Laplacian polynomial of \( K_{c_n}(k) \) [4].

9. If \( k = 1 \) and \( p = 3 \) then equation (1) reduces to

\[ C(K_{d_n}(3, 1)) = \mu(\mu - n)^{n-3}(\mu - n + 3)^2 \] which is a Laplacian polynomial of \( K_{d_n}(3) \).

REFERENCES


AUTHOR’S BIOGRAPHY

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