On Left Quasi Noetherian Rings

Falih A.M. Aldosray
Department of mathematics
Umm AlQura University,
Makkah, P o Box 56199, Saudi Arabia
fadosary@uqu.edu.sa

Amani M. A. Alfadli
Department of mathematics
Umm AlQura University
Makkah, P.O.Box 56199 Saudi Arabia
tala5353@hotmail.com

Abstract: In this paper we prove; If $R$ is a left quasi-Noetherian ring ,then every nil subring is nilpotent). Next we show that a commutative semi-prime quasi-Noetherian ring is Noetherian. Then we study the relationship between left Quasi-Noetherian and left Quasi-Artinian, in particular we prove that If $R$ is a non-nilpotent left Quasi-Artinian ring. Then any left $R$-module is left Quasi-Artinian if and only if it is left Quasi-Noetherian. Finally we show that a commutative ring $R$ is Quasi-Artinian if and only if $R$ is Quasi-Noetherian and every proper prime ideal of $R$ is maximal.

Keywords: Noetherian and Artinian, Left Quasi-Noetherian and Left Quasi-Artinian Rings.

1. INTRODUCTION

By a ring we mean an associative ring that need not have an identity. Following [1] we saythat a left $R$-Module $M$ is left quasi-Noetherian if for every ascending chain $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ of $R$-submodules of $M$,there exists $m \in \mathbb{Z}^+$ such that $Rm \cap (\bigcup_{i \in M} N_i) \subseteq N_m$. We say that the ring $R$ is a left quasi-Noetherian ring if $R$ is quasi-Noetherian. Note that any left Noetherian ring or module is a left quasi-Noetherian. Also any nilpotent ring is a left quasi-Noetherian, however $R = \left[ \begin{array}{cc} \mathbb{Z} & 0 \\ 0 & \mathbb{Q} \end{array} \right]$ is a non-nilpotent ring which is a left quasi-Noetherian but not Noetherian.

Note that : if $M$ is a left quasi-Noetherian module and $N$ is a submodule of $M$, then $M/N$ is a left quasi-Noetherian[1. Proposition 1.3]

Proposition 1.1:
Let $R$ be a left quasi-Noetherian, $I \triangleleft R$. Then $I$ is a left quasi-Noetherian.

Proof:
Let $J_1 \subseteq J_2 \subseteq \cdots$ be any ascending chain of left ideals of $I$, then $I_1 \subseteq J_2 \subseteq \cdots$ is an ascending chain of left ideals of $R$. But $R$ is a left quasi-Noetherian so there exists $m \in \mathbb{Z}^+$ such that $I^m \cap (\bigcup_{i \in M} J_i) \subseteq J_m \subseteq J_{m+1}$. Hence $I$ is a left quasi-Noetherian.

Now : If $I \triangleleft R$ and $I$, $R/I$ are left quasi-Noetherian then $R$ need not be a left quasi-Noetherian ring, as the following example shows: Let $R = \left[ \begin{array}{cc} \mathbb{Z} & 0 \\ 0 & \mathbb{Q} \end{array} \right]$, $I = \left[ \begin{array}{cc} 0 & \mathbb{Q} \\ 0 & 0 \end{array} \right] \triangleleft R$, hence $I$ and $R/I = \mathbb{Z} \oplus \mathbb{Q}$ are left quasi-Noetherian but $R$ is not, however we can prove the following :

Let $I \triangleleft R$, then $R$ is a left quasi-Noetherian if one of the following holds:

(a) $R/I$ is a left quasi-Noetherian and if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots \subseteq I$ and $I_1 \triangleleft R$ then there exists $m \in \mathbb{Z}^+$ such that $R^m \cap (\bigcup_{i \in M} I_i) \subseteq I_m$.

(b) $R/I$ is a left quasi-Noetherian and $I$ is a left Noetherian.

Proof:
(a) Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be an ascending chain of left ideals of $R$. Then $I_1 \cap I \subseteq I_2 \cap I \subseteq \cdots \subseteq I_n \cap I \subseteq \cdots \subseteq I$ then there exist $m \in \mathbb{Z}^+$ such that $R^m \cap (\bigcup_{i \in M} I_i) \subseteq I_m \cap I$. Also $\frac{I_1}{I} \subseteq \frac{I_2}{I} \subseteq \cdots \subseteq \frac{I_n}{I} \subseteq \cdots$ is an ascending chain of left ideals of $R/I$. But $R/I$ is a left quasi-
Noetherian ring so there exists \( m \in \mathbb{Z}^+ \) such that \((R/I)^m(U_n \frac{I_m}{I}) \subseteq \frac{I_m^{t+1}}{I}\) which implies that \(R^m(U_n l_n + I) \subseteq l_m + I\). Now \(R^m(U_n l_n) \subseteq R^m(U_n l_n + I) \cap (U_n l_n) \subseteq (l_m + I) \cap (U_n l_n) = (U_n l_n \cap I) + l_m\) so \(R^m(U_n l_n) \subseteq R^m((U_n l_n \cap I) + l_m) = R^m(U_n l_n \cap I) + R^m l_m \subseteq (l_m \cap I) + l_m = l_m\). Therefore \(R^{2m}(U_n l_n) \subseteq l_m \subseteq l_{2m}\). Hence is a left quasi-Noetherian.

(b) Can be prove by the same way.

**Proposition 1.3:**

A finite direct sum of left quasi-Noetherian rings is a left quasi-Noetherian.

**Proof:**

By induction, it is enough to prove the result for \( t = 2 \). So let \( R_1 \oplus R_2 \) , \( R_1 \), \( R_2 \) are left Quasi-Noetherian. Now let \( I_1 \subseteq l_2 \subseteq \cdots \subseteq l_n \subseteq \cdots \) be any ascending chain of left ideals of \( R \). Then \( R_1 I_1 \subseteq R_1 l_2 \subseteq \cdots \subseteq R_1 l_n \subseteq \cdots \) is an ascending chain of left ideals of \( R_1 \) and \( R_2 I_1 \subseteq R_2 l_2 \subseteq \cdots \subseteq R_2 l_n \subseteq \cdots \) is an ascending chain of left ideals of \( R_2 \). But \( R_1 \) and \( R_2 \) are left Quasi-Noetherian rings, therefore there exists \( m \in \mathbb{Z}^+ \) such that \( R_1^m(U_n R_1 l_n) \subseteq R_1 l_m \subseteq l_m \) and \( R_2^m(U_n R_2 l_n) \subseteq R_2 l_m \subseteq l_m \). Hence \( R^{m+1}_1(U_n l_n) \subseteq R_1^m(U_n R_1 l_n) + R_2^m(U_n R_2 l_n) \subseteq l_m \subseteq l_{m+1} \). Therefore \( R \) is a left Quasi-Noetherian ring.

An ideal \( Q \) in a ring \( R \) is said to be a semi-prime ideal if and only if \( A^2 \subseteq Q \), \( A < R \), then \( A \subseteq Q \), it follows easily by induction that if \( Q \) is a semi-prime ideal in \( R \) and \( A^2 \subseteq Q \) for an arbitrary positive integer \( n \), then \( A \subseteq Q \{15, P.67\} \)

A ring \( R \) is said to be regular if for each element \( a \in R \) there exist some \( a' \in R \) such that \( aa'a = a \). Note that a commutative ring \( R \) is regular if and only if every ideal of \( R \) is semiprime [5, P.186].

By the nil radical \( N=N(R) \) of a ring \( R \) we mean the sum of all nilpotent ideals of \( R \), which is a nil ideal. It is well known [10. P.28. Theorem 2], that \( N \) is the sum of all nilpotent left ideals of \( R \) and it is the sum of all nilpotent right ideals of \( R \).

A ring \( R \) is said to be a left Goldie ring if:

(a) \( R \) satisfies the a.c.c on left annihilator ideals. (b) \( R \) has no infinite direct sum on left ideals. We can prove the following:

**Proposition 1.4:**

If \( R \) is a left quasi-Noetherian ring and \( r(R) = 0 \), then \( R \) is a left Goldie ring.

**Proof:**

First we show that any ascending chain of left annihilator idealterminates. Let \( I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots \) be any ascending chain of left annihilator ideals of \( R \). Suppose that \( I_i = l(I_i) \) for all \( i \). Since \( R \) is a left quasi-Noetherian ring then there exists \( m \in \mathbb{Z}^+ \) such that \( R^m(U_n l_n) \subseteq l(I_m) \), therefore \( R^m(U_n l_n)l_m = 0 \), and \( R^{m-1}(U_n l_n)l_m = 0 \) But \( r(R) = 0 \), hence \( R^{m-1}(U_n l_n)l_m = 0 \). Continuing in this way we have \( R(U_n l_n)l_m = 0 \), therefore \( (U_n l_n)l_m = 0 \), and \( U_n l_n \subseteq l(I_m) = l_m \). Hence \( I_m = I_{m+1} = \cdots \) and the chain terminates.

Now let \( l_1 \subseteq l_2 \subseteq \cdots \subseteq l_n \subseteq \cdots \) be any ascending chain of complement left ideals of \( R \). Since \( R \) is a left Quasi-Noetherian ring then there exists \( m \in \mathbb{Z}^+ \) such that \( R^m(U_n l_n) \subseteq l(I_m) \). Now suppose that \( I_m \) is a complement of \( J_m \), then \( R^m(U_n l_n) \cap J_m \subseteq l(I_m) = J_m \). But \( (U_n l_n) \cap J_m \subseteq U_n l_n \) and \( (U_n l_n) \cap I_m \subseteq I_m \). Hence \( R^m((U_n l_n) \cap J_m) \subseteq R^m(U_n l_n) \) and \( R^m(U_n l_n) \cap J_m \subseteq (U_n l_n) \cap J_m \subseteq l_m \). Therefore \( R^m(U_n l_n) \cap J_m \subseteq (R^m(U_n l_n)) \cap J_m = 0 \) and \( R(R^{m-1}(U_n l_n)) = 0 \). But \( r(R) = 0 \), hence \( R^{m-1}(U_n l_n) \cap J_m = 0 \). Continuing in this way we have \( (U_n l_n) \cap J_m = 0 \), and by maximality of \( I_m \) we have \( U_n l_n = l_m \). Hence \( I_m = I_{m+1} = \cdots \). Therefore \( R \) is a left Goldie ring.
Following [2] we say that a left $R$-Module $M$ is left quasi-Artinian if for every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ of $R$-submodules of $M$, there exists $m \in \mathbb{Z}^+$ such that $R^m N_m \subseteq N_n$ for all $n$. We say that the ring $R$ is a left quasi-Artinian ring if $R$ is quasi-Artinian. Now we prove the following:

**Proposition 1.5:**
Any semi-prime left quasi-Artinian ring is a semi-simple left Artinian.

**Proof:**
By [2, Theorem 2.4] every non-zero left ideal of $R$ is generated by a non-zero idempotent $e$, say. But we know that $e$ acts as right identity for the left ideal $I = Re$, and since $R$ is itself an ideal, hence $R$ has an identity element. Therefore $R$ is left Artinian. Now, $J(R)$ is nilpotent, and $R$ is a semi-prime ring, implies that $J(R) = 0$. Hence $R$ is a semi-simple.

2.
In this section we prove the following:

**Theorem 2.1:**
Let $R$ be a left quasi-Noetherian ring. Then every nil subring of $R$ is nilpotent.

**Proof:**
Since $R \supseteq R^2 \supseteq \cdots \supseteq R^n \supseteq \cdots$, it follows that $r(R) \subseteq r(R^2) \subseteq \cdots \subseteq r(R^n) \subseteq \cdots$ is an ascending chain of ideals of $R$. But $R$ is a left quasi-Noetherian ring hence there exists $m \in \mathbb{Z}^+$ such that $R^m r(R^t) \subseteq r(R^{tm})$ for all $t$. Therefore $R^{2m} r(R^{2m}) \subseteq R^{m} r(R^{m}) = 0$, and $r(R^t) \subseteq r(R^{2m})$ so that $r(R/r(R^{2m})) = 0$. But $\overline{R} = R/r(R^{2m})$ is a left quasi-Noetherian hence $\overline{R}$ is a left Goldie ring. By Lanski Theorem [14] any nil subring $\overline{S}$ of $\overline{R}$ is nilpotent so there exists $n \in \mathbb{Z}^+$ such that $\overline{S}^n = 0$ and then $S^n \subseteq r(R^{2m})$ so $S^{n+2m} = 0$. Hence $S$ is nilpotent subring of $R$.

An immediate consequence we have the following;

**Corollary 2.2:**
Let $R$ be a left quasi-Noetherian ring, then $N(R)$ is nilpotent.

**Theorem 2.3:**
If $R$ is a left quasi-Noetherian ring. Then $R$ satisfies the ascending chain condition on semi-prime ideals.

**Proof:**
Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be any ascending chain of semi-prime ideals of $R$. Then there exists $m \in \mathbb{Z}^+$ such that $R^m (U_n I_n) \subseteq I_m$. But $U_n I_n \subseteq R$, hence $(U_n I_n)^m \subseteq R^m$ and $(U_n I_n)^{m+1} = (U_n I_n)^m (U_n I_n) \subseteq R^m (U_n I_n) \subseteq I_m$. But $I_m$ is a semi-prime ideal, hence $(U_n I_n) \subseteq I_m$ so $I_m = I_{m+1} = \cdots$.

**Corollary 2.4:**
If $R$ is a commutative regular quasi-Noetherian ring. Then $R$ is Noetherian.

**Proof:**
Since $R$ is a commutative regular ring it follows that every ideal of $R$ is semi-prime. But $R$ is quasi-Noetherian hence by (Theorem 2.3) $R$ is Noetherian ring.

**Theorem 2.5:**
Let $R$ be a commutative semi-prime quasi-Noetherian ring. Then $R$ is Noetherian.

To prove this we need the following lemma

**Lemma 2.6:**
If $R$ is a left quasi-Noetherian ring so $R$ has a finite number of minimal prime ideals of $R$. 

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Proof:
By [1, Corollary 3.8] there exists a finite number of prime ideals \( P_1, P_2, \ldots, P_n \) of \( R \) such that \( \prod_{i=1}^{n} P_i = 0 \). Now let \( P \) be any minimal prime ideal of \( R \) so \( \prod_{i=1}^{n} P_i \subseteq P \) therefore \( P_1 \subseteq P \) for some \( i \) but \( P \) is minimal so \( P = P_t \) hence there exists a finite number of minimal prime ideals of \( R \).

Proof of Theorem 2.5:
Let \( P_t \) be a minimal prime ideal in \( R \), \( P_2 \) is a minimal prime ideal of \( P_1 \) (isolated prime of \( R \)) so \( \exists \) (is maximal prime ideal in \( R \)) continuing in this way we have \( P_1 \) is a maximal prime ideal in \( P_2 \) so \( P_2/P_1 \) contains no non-zero prime ideal, therefore every factor of \( P_2/P_1 \) (Otherwise if \( T = P_2/P_1 \) and \( \bar{T} = T/I, I \triangleleft T \), has a non-zero prime ideal say \( \bar{J} \) so \( \pi^{-1}(\bar{J}) \) is a prime ideal in \( T \) where \( \pi: T \to T/I \) is a natural homomorphism, which mean that \( \pi^{-1}(\bar{J}) = 0 \) then \( \bar{J} = \pi(\pi^{-1}(\bar{J})) = \pi(0) = 0 \). Hence \( T \) has no non-zero prime ideal). Therefore every factor of \( P_2/P_1 \) is a quasi-Noetherian. Hence by Proposition 1.4 every factor of \( P_2/P_1 \) is a Goldie ring and by Camilo’s Theorem \( P_2/P_1 \) is a Noetherian ring.

Now \( R = P_t \oplus R/P_t, P_t = P_{t-1} \oplus P_t/P_{t-1} \). \( P_{t-1} \) is maximal prime ideal in \( P_t \) and so on.

Therefore \( R = P_1 \oplus P_2/P_1 \oplus P_3/P_2 \oplus \cdots \oplus P_t/P_{t-1} \oplus R/P_t \) and \( R/P_t, P_t/P_{t-1} \) for all \( t = 1, \ldots, t \) are Noetherian. Hence \( R/P_1 \cong P_2/P_1 \oplus P_3/P_2 \oplus \cdots \oplus P_t/P_{t-1} \oplus R/P_t \) is a finite direct sum of Noetherian rings so it is Noetherian. By Lemma 2.6 \( R \) has a finitenumber of minimal prime ideals therefore \( N(R) = \bigcap_{i=1}^{n} P_i \) is the minimal prime ideal in \( R \), but \( R \) is a semi-prime ring hence \( N(R) = 0 \) and \( R \cong R/N(R) \oplus \bigoplus_{i=1}^{n} R/P_i \) is Noetherian.

3.

In this section we study the relationship between left quasi-Noetherian and left quasi-Artinian. In particular we prove the following:

Theorem 3.1:
If \( R \) is a non-nilpotent left quasi-Artinian ring. Then any left \( R \)-module is a left quasi-Artinian if and only if it is a left quasi-Noetherian.

Proof:
Since \( R \) is a non-nilpotent it follows that \( R \neq N(R) \). But \( R \) is a left quasi-Artinian, hence the nil radical \( N(R) \) is nilpotent. Therefore \( N^t = 0 \) for some \( t \). Now let \( R \)-module be any left quasi-Artinian left \( R \)-module. This has a chain of submodules \( M \supseteq N \supseteq N^2 \supseteq \cdots \supseteq N^t \) which factor modules \( F_k = N^{k-1}M/N^k M, k = 1, \ldots, t \). Now \( F_k \) is annihilated by \( N \) hence maybe regarded as an \( R/N \)-module. Since \( R \) is a left quasi-Artinian ring so \( R/N \) is a semi-prime left quasi-Artinian and by [2, Theorem ] \( R/N \) is a semi-simple Artinian so by [15, proposition 2, pg 68] \( R/N \) is completely reducible, hence \( F_k \) is completely reducible as an \( R/N \)-module and therefore also as an \( R \)-module. Since \( F_k \) is a unital left quasi-Artinian \( R/N \)-module so \( F_k \) is a left Artinian as an \( R/N \)-module then \( F_k \) is the direct sum of finite number of irreducible \( R \)-modules, hence \( F_k \) is Noetherian and then left quasi-Noetherian. Thus \( F_t = N^{t-1}M/N^t M = N^{t-1}M \) and \( F_{t-1} = N^{t-2}M/N^{t-1}M \) are left quasi-Noetherian, hence so is \( N^{t-2}M \). Continuing in this way we have \( M \) is a left quasi-Noetherian \( R \)-module.

To prove the converse replace \( R \)-module instead of Artinian

Theorem 3.2:
Let \( R \) be a commutative ring. Then \( R \) is quasi-Artinian if and only if \( R \) is quasi-Noetherian and every proper prime ideal of \( R \) is maximal.

Proof:
Since $R$ is commutative so $(R) = \text{rad}(R) = \bigcap_i P_i$, $P_i$ is minimal prime ideal of $R$, $\text{rad}(R)$ denoted the prime radical of $R$. Let $R$ be a quasi-Noetherian ring so by (Lemma 2.6) $R$ has a finite number of minimal prime ideals of $R$ so $N(R) = \bigcap_{i=1}^n P_i$.

Now $R \cong N(R) \oplus R/N(R)$ but $N(R/N(R)) = \overline{0}$ so $\overline{R} \cong R/N(R) \cong \bigoplus_{i=1}^n \overline{R}/\overline{P_i}$ and $\overline{R}/\overline{P_i}$ prime ring. Since every prime ideal of $R$ is maximal so also in $\overline{R} = R/N(R)$ then each of $\overline{P_i}$ is maximal ideal in $\overline{R}$ therefore $\overline{R}/\overline{P_i}$ simple rings so quasi-Artinian and hence $\overline{R} = R/N(R)$ is a quasi-Artinian ring. Since $N(R)$ is nil ideal of $R$ and $R$ is a quasi-Noetherian ring so $N(R)$ is nilpotent then quasi-Artinian hence $R$ is quasi-Artinian ring.

To prove the converse let $R$ be a quasi-Artinian ring, $R \cong N(R) \oplus R/N(R)$, $N(R)$ is nilpotent ring so quasi-Noetherian ring and $R/N(R)$ is a semi-prime quasi-Artinian ring so it is a semi-simple Artinian ring therefore quasi-Noetherian and hence $R$ is a quasi-Noetherian ring.

REFERENCES


