International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 2, February 2014, PP 165-170
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Generalised lattice ordered groupoids (gl – groupoids)

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Abstract: *This paper, deals with the concept of gl-groupoid in such a way that every l-groupoid is a gl-groupoid but not conversely. The concepts of gl-semigroup, gl-monoid, residuated generalised lattice and integral gl-groupoid have been identified.. and also Galois connection on a residuated generalised lattice is established . Proved that a complemented integral gl-groupoid satisfying some condition is distributive.*

Keywords: *poset, lattice, groupoid, po-groupoid, l-groupoid.*

1. Introduction

Murty and Swamy [3] introduced the concept of generalised lattice. In [4], the author developed the theory of generalised lattices by proving several results in generalised lattices which are true in case of lattices and those are unable to prove in posets. The theory of lattice ordered groupoids (l-groupoids) is well known by chapter XIIV of [1]. In this paper section 1 contains preliminaries which are taken from the references [1], [3], [5] and [6]. In section 2, introduced the concept of generalised lattice ordered groupoid (gl-groupoid) in such a way that every l-groupoid is a gl-groupoid and gave an example of a gl-groupoid which is not an l-groupoid. Similarly introduced the concepts of gl-semigroup, gl-monoid and integral gl-groupoid and proved some useful results. In section 3 introduced the concept of residuated generalised lattice and obtained a Galois connection on a residuated generalised lattice. In section 4 studied about integral gl-groupoids and proved that a complemented integral gl-groupoid satisfying some condition is distributive.

2. Preliminaries

2.1. Definition: [3] Let (P, ≤) is a poset. Then P is said to be a generalised meet semilattice if for every non empty finite subset A of P, there exist a non empty finite subset B of P such that, $x \in$ L(A) if and only if $x \leq b$ for some $b \in$ B. P is said to be a generalised join semilattice if for every non empty finite subset A of P, there exist a non empty finite subset B of P such that, $x \in$ U(A) if and only if $b \leq x$ for some $b \in$ B. P is said to be a generalised lattice if it is both generalised meet and join semilattice.

If P is a generalised meet semilattice, then for any L(A) ∈ $\mathcal{L}$ (P) there exists a unique finite subset B of P such that L(A) = $\bigcup_{b \in B}$ L(b) and the elements of B are mutually incomparable and the set is denoted by ML(A). Similarly observed that: If P is a generalised join semilattice, then for any U(A) ∈ A (P) there exists a unique finite subset B of P such that U(A) = $\bigcup_{b \in B}$ U(b) and the elements of B are mutually incomparable and the set is denoted by mu(A).

2.2. Definition [3] An element a of a poset P is said to be meet distributive if L(a) is meet distributive in the semilattice $\mathcal{L}$ (P), it is said to be join distributive if U(a) is meet distributive in the semilattice A (P), it is said to be distributive if it is both meet and join distributive. A poset P is said to be meet(join) distributive if every element of P is meet(join) distributive. P is said to be distributive if every element of P is distributive. A poset P is a meet (join) distributive poset if and only if $\mathcal{L}$ (P) (A (P)) is a distributive meet semilattice. If P is a generalised lattice then $\mathcal{L}$ (P), A

(P) are lattices. A generalised lattice P is a distributive if and only if L (P) (A (P)) is distributive.

2.3. Definition [5] Let P be bounded generalised lattice with least element 0 and greatest element 1. An element $a \in$ P is said to be complemented if there exists $b \in$ P such that $a \wedge b$ exists and equal to 0 and $a \vee b$ exists and equal to 1. If every element of P is complemented then P is said to be complemented.

2.4. Definition [6] A system (G, +, ≤) is called a gl-group (= generalized lattice ordered group) if (G, ≤) is a generalised lattice, (G, +) is a group and every group translation $x \to a + x + b$ on G is isotone (that is $x \leq y \Rightarrow a + x + b \leq a + y + b$ for all $a, b \in$ G).

2.5. Definition [1] A po-groupoid (or m-post) is a poset M with a binary multiplication which satisfies the isotonicity condition (1): $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for all $a, b, x \in$ M.

2.6. Definition [1] Let M be a po-groupoid. The right-residual $a \therefore b$ of a by b is the largest x (if it exists) such that $bx \leq a$. The left-residual $a \because b$ of a by b is the largest y (if it exists) such that $yb \leq a$.

3. GL-GROUPOIDS

3.1. Definition: A multiplicative generalised semilattice (or m-gsl) is a generalized (join) semilattice M with multiplication such that, for any $a, b, c \in$ M satisfying the conditions,

(2): mu{b, c} = {$z_1, z_2, \ldots, z_n$} implies mu{ab, ac} = {$az_1, az_2, \ldots, az_n$} and

(3): mu{a, b} = {$z_1, z_2, \ldots, z_n$} implies mu{ac, bc} = {$z_1c, z_2c, \ldots, z_nc$}.

3.2. Definition: A generalised lattice M with a multiplication satisfying (2) and (3) is called a multiplicative generalised lattice (m-gl) or gl-groupoid.

3.3. Definition: A gl-groupoid which is a semi group (monoid) under the multiplication is called a gl-semigroup (gl-monoid).

(1) follows from (2) & (3), (for if $b \leq c$, then mu{ab, ac} = {ac} and mu{ba, ca} = {ca}, whence $ab \leq ac$ and $ba \leq ca$). In other words, any m-gsl is a po-groupoid. Since every directed below finite poset is a generalised meet semilattice, any finite m-gsl with least element 0 is a gl-groupoid. Consequently, if G is any gl-monoid with identity element e then $ax \geq x$ for all $x \in$ G if and only if $a \geq e$ while, $ax \leq x$ for all $x \in$ G if and only if $a \leq e$.

Any po-group which is a generalised lattice satisfies (2), (3) and their dual. In other words, every gl-group is a gl-monoid. Every l-groupoid is a gl-groupoid but the converse is not true. Similarly every l-semigroup (l-monoid) is a gl-semigroup (gl-monoid). Any chain is a gl-groupoid as well as an l-groupoid under any isotone multiplication.

3.4. Example: Consider a generalised lattice M (see fig(1)). Define a multiplication on M such that $xy = x$ for all $x, y \in$ M. Then clearly M is a gl-groupoid but not an l-groupoid and M is a gl-semigroup but not an l-semigroup.

3.5. Example: Let M be a bounded generalised lattice with least element 0 and greatest element 1. Suppose M^1 = M ∪ {e} where $0 < e < 1$ and e is incomparable to x for all $x \in$ M - {0, 1}. Then M^1 is a complemented generalised lattice (see fig(2)). Define a multiplication on M^1 such that $xy = x$ for all $x, y \in$ M and $ex = xe = x$ for all $x \in$ M. Then clearly M^1 is a gl-monoid but not an l-monoid.

3.6. Definition Let M be a gl-groupoid with identity element e. An element a of M is said to integral if $a \leq e$. M is said to be integral gl-groupoid if $x \leq e$ for all $x \in$ M.

3.7. Example: Let M be a generalised lattice and $M^1 = M \cup \{e\}$ where $x \le e$ for all $x \in$ M. Then M^1 is a generalised lattice with greatest element e (see fig(3)). Define a multiplication on M^1 such that $xy = x$ for all $x, y \in$ M and $ex = xe = x$ for all $x \in$ M. Then M^1 is an integral gl-monoid.

3.8. Lemma In any gl-groupoid, we have $U(\{ba, ab\}) \subseteq \bigcap_{s \in ML\{a,b\}} \bigcup_{t \in mu\{a,b\}} U(st)$.

Proof: Let $ML\{a,b\} = \{s_1, s_2, \ldots, s_n\}$ and $mu\{a,b\} = \{t_1, t_2, \ldots, t_m\}$. Then by (1) we have $s_i a \le ba$ and $s_i b \le ab$ for all i. This gives $U(\{ab, ba\}) \subseteq \bigcap_{i=1}^{n} U(\{s_i a, s_i b\})$ and by (2) we have $U(\{s_i a, s_i b\}) = \bigcup_{j=1}^{m} U(s_i t_j)$. Therefore $U(\{ba, ab\}) \subseteq \bigcap_{i=1}^{n} \bigcup_{j=1}^{m} U(s_i t_j)$.

3.9. Lemma In any integral gl-groupoid, we have

(i) $mu\{a,b\} = \{e\}$ implies $U(ML\{a,b\}) = U(\{ba, ab\})$ and

(ii) $mu\{a,b\} = mu\{a,c\} = \{e\}$ implies $mu\{a, bc\} = \{e\}$ and $mu\{a,s\} = \{e\}$ for some $s \in ML\{b,c\}$.

Proof: i): Observe that in an integral gl-groupiod, $U(ML\{a,b\}) \subseteq U(\{ba, ab\})$. If $mu\{a,b\} = \{e\}$ then by above lemma we get $U(\{ab, ba\}) \subseteq U(ML\{a,b\})$ and therefore $U(ML\{a,b\}) = U(\{ba, ab\})$. ii): Observe that $U(e) \subseteq U(\{a, bc\}) \subseteq U(\{aa, ba, ac, bc\})$. Suppose $mu\{a,b\} = mu\{a,c\} = \{e\}$. Then by (2) we have $U(\{aa, ba, ac, bc\}) = U(\{a,b\}) = U(e)$ and therefore $U(\{a, bc\}) = U(e)$ or $mu\{a, bc\} = \{e\}$. Observe that $U(e) \subseteq U(\{a,s\}) \subseteq U(\{a, bc\}) = U(e)$ for some $s \in ML\{b,c\}$. Therefore $mu\{a,s\} = \{e\}$ for some $s \in ML\{b,c\}$.

4. RESIDUATED GENERALISED LATTICES

4.1. Definition A residuated generalised lattice is a gl-groupiod L in which $a \therefore b$ and $a \because b$ exists for any $a, b \in$ L.

Observe that any gl-groupoid which satisfies ascending chain condition is residuated, provided the inequalities $xa \le b$ and $ay \le b$ have solutions x, y for all a, b.

In any po-groupoid, the functions $a \therefore b$ and $a \because b$ are isotone in a and antitone in b. This gives the third condition of the following theorem.

4.2. Theorem In any residuated generalised lattice we have

(i) $ML\{a,b\} = \{s_1, s_2, \ldots, s_n\}$ implies $ML\{a \therefore c, b \therefore c\} = \{s_1 \therefore c, \ldots, s_n \therefore c\}$

(ii) $mu\{b,c\} = \{s_1, s_2, \ldots, s_n\}$ implies $ML\{a \therefore b, a \therefore c\} = \{a \therefore s_1, \ldots, a \therefore s_n\}$

(iii) $s \in ML\{b,c\}$ implies $a \therefore s \in U(\{a \therefore b, a \therefore c\})$.

Proof: (i): Let $a \therefore c = y$, $b \therefore c = z$ and $s_i \therefore c = x_i$ for $1 \le i \le$ n. Let $ML\{y,z\} = \{t_1, t_2, \ldots, t_m\}$. Consider an index i ($1 \le i \le$ n). Then $x_i \le t_j$ for some j ($1 \le j \le$ m). But $ct_j \in L(\{a,b\})$ gives $ct_j \le s_k$ for some k ($1 \le k \le$ n). Therefore $x_i \le t_j \le x_k$. Finally we can say that each x_i is equal to t_j for some j and conversely. Hence $ML\{a \therefore c, b \therefore c\} = ML\{y,z\} = \{t_1, t_2, \ldots, t_m\} = \{x_1, x_2, \ldots, x_n\} = \{s_1 \therefore c, \ldots, s_n \therefore c\}$. (ii): Let $a \therefore c$ = y, $b \therefore c$ = z and $a \therefore s_i = x_i$ for $1 \le i \le$ n. Let $ML\{y,z\} = \{t_1, t_2, \ldots, t_m\}$. Consider an index i $(1 \le i \le$ n). Then $s_i t_j \le$

a for some j and therefore $t_i \leq x_j$. Since $b, c \leq s_j$, we have $bx_j \leq a$ and $cx_j \leq a$. This implies $x_j \in$ L({ y, z }) and then $x_j \leq t_k$ for some k ($1 \leq k \leq$ m). Therefore $t_i \leq x_j \leq t_k$. Finally we can say that each t_i is equal to x_j for some j and conversely. Hence ML{ $a \iddots c$, $b \iddots c$ } = ML{ y, z } = { $t_1, t_2, \ldots, t_m$ } = { $x_1, x_2, \ldots, x_n$ } = { $a \iddots s_1$, …, $a \iddots s_n$ }.

Observe that in any residuated generalised lattice, we have $b \leq a \iddots (a \ddots b)$ and $b \leq a \ddots a \iddots b$). Any integral element a of a residuated generalized lattice with unity e satisfies $a \leq e \iddots (e \ddots a) \leq$ e and $a \leq e \ddots (e \iddots a) \leq e$.

4.3. Theorem For any fixed element c of any residuated generalised lattice L, the correspondences $x \rightarrow c \iddots x = x^*$ and $y \rightarrow c \ddots y = y^+$ define a Galois connection on L.

4.4. Definition Let L be a residuated generalised lattice and $c \in$ L**.** An element $x \in$ L is said to be a right c-closed if $x = c \iddots (c \ddots x)$. An element $x \in$ L is said to be left c-closed if $x = c \ddots (c \iddots x)$.Observe that in a residuated lattice L, an element x is right c-closed if and only if $x = c \iddots y$ for some y and left c-closed if and only if $x = c \ddots y$ for some y. More over if a, b are right c-closed elements then so is every element of ML{ a, b }.

5. INTEGRAL GL-GROUPOIDS

5.1. Definition Any pair of elements a, b of an integral gl-groupoid L are said to be coprime if mu{ a, b } = { e }.

Observe that in an integral gl-groupoid, for any x, y we have U(ML{ x, y }) $\subseteq$ U(xy).

5.2. Lemma In an integral gl-groupoid, if a, b are coprime then mu{ xa, xb } = { x } = mu(ML{ x, a } $\cup$ ML{ x, b }) for all x.

Proof: Suppose mu{ a, b } = { e }. Then by definition clearly mu{ xa, xb } = { xe } = { x }. Let mu(ML{ x, a } $\cup$ ML{ x, b }) = { $f_1, f_2, \ldots, f_k$ }. Then each f_i is an upper bound of { xa, xb } and therefore $x \leq f_i$ for all i. On the other hand, clearly $x \in$ U(ML{ x, a } $\cup$ ML{ x, b }) and then $x \geq f_j$ for some j. Therefore $x = f_i$ for all i.

5.3. Lemma In an integral gl-groupoid, if a, b are coprime and x ∈ U(ML{ a, b }) then ML(mu{ x, a } $\cup$ mu{ x, b }) = { x } = mu(ML{ x, a } $\cup$ ML{ x, b }).

Proof: Suppose mu{ a, b } = {e} and $x \in$ U(ML{ a, b }). Let mu{ x, a } = { $c_1, c_2, \ldots, c_m$ }, mu{ x, b } = { $d_1, d_2, \ldots, d_n$ } and ML(mu{ x, a } $\cup$ mu{ x, b }) = { $f_1, f_2, \ldots, f_r$ }. Then for each i ($1 \leq i \leq$ r) we have $bf_i \leq bc_j$ and $af_i \leq ad_k$ for all j, k ($1 \leq j \leq$ m, $1 \leq k \leq$ n). This gives U({ bx, ba }) = $\bigcap_{j=1}^{m}$ U(bc_j) $\subseteq$ U(bf_i) and U({ ax, ab }) = $\bigcap_{k=1}^{n}$ U(ad_k) $\subseteq$ U(af_i). Therefore U({ ax, ab, bx, ba }) $\subseteq$ U(f_i) for all i. Now we can observe that U(ML{ a, x }) $\subseteq$ U({ ax, ab }) and U(ML{ b, x }) $\subseteq$ U({ bx, ba }). This implies and by above lemma we have U(x) = U(ML{ x, a } $\cup$ ML{ x, b }) $\subseteq$ U(f_i) and therefore $f_i \leq x$ for all i. On the other hand since $x \in$ L(mu{ x, a } $\cup$ mu{ x, b }), $x \leq f_j$ for some j. Therefore finally $x = f_i$ for all i. That is ML(mu{ x, a } $\cup$ mu{ x, b }) = { x }.

5.4. Lemma In an integral gl-groupoid, if x, y are both coprime to a then so are xy and p for some $p \in$ ML{ x, y }.

Observe that if a generalised lattice P is integral then L (P) is also integral (i.e., $a \leq e$ for all $a \in$ P implies L(A) ≤ L(e) for all L(A) ∈ L (P)).

5.5. Theorem In an integral gl-groupoid, for any coprime elements a, b we have,

(s, t) ∈ [a, e] × [b, e] implies mu(ML{ s, t } ∪ { a }) = { s } and mu(ML{ s, t } ∪ { b }) = { t }.

Proof: Since a, b are coprime and b ≤ t; the elements t, a are also coprime. Then by Lemma 4.2 we have mu(ML{ s, t } ∪ { a }) = mu(ML{ s, t } ∪ ML{ s, a }) = { s }. Similarly we can prove mu(ML{ s, t } ∪ { b }) = { t }.

In an integral gl-groupoid P, consider the following condition (4): a, b are coprime and L(X) ∈ [L({ a, b }), L(e)] implies $\bigcap_{x \in X}$ (L(x) ∨ L(a)) = L(X) ∨ L(a) and $\bigcap_{x \in X}$ (L(x) ∨ L(b)) = L(X) ∨ L(b).

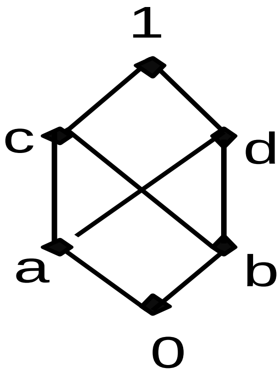
5.6. Theorem If P is an integral gl-groupoid satisfying the condition (4) then for any coprime elements a, b ; we have [L({ a, b }), L(e)] is isomorphic to [L(a), L(e)] × [L(b), L(e)] in L (P).

Proof: Define a map φ: [L({ a, b }), L(e)] → [L(a), L(e)] × [L(b), L(e)] by φ(L(X)) = (L(X) V L(a), L(X) V L(b)). Clearly φ is well defined. To show that φ is onto: Let (L(Y), L(Z)) ϵ [L(a), L(e)] × [L(b), L(e)]. Then L(Y∪ Z) = L(Y) ∩ L(Z) ϵ [L({ a, b }), L(e)]. Now for any y ∈ Y and z ∈ Z, we have (y, z) ∈ [a, e] × [b, e] and then by above theorem, mu(ML{ y, z } ∪ { a }) = { y } and mu(ML{ y, z } ∪ { b }) = { z }. Therefore L({ y, z }) V L(a) = L(y) and L({ y, z }) V L(b) = L(z). By the condition (4), L(Y∪ Z) V L(a) = $\bigcap_{y \in Y} \bigcap_{z \in Z}$ (L({ y, z }) V L(a)) = $\bigcap_{y \in Y} \bigcap_{z \in Z}$ L(y) = L(Y). Similarly L(Y∪ Z) V L(b) = L(Z). To show that φ is one-one: Let L(S), L(T) ∈ [L({ a, b }), L(e)] and suppose φ(L(S)) = φ(L(T)). Now by condition (4) and Lemma 4.3 we have (L(S) ∨ L(a)) ∧ (L(S) ∨ L(b)) = $\bigcap_{s \in S}$ ((L(s) ∨ L(a)) ∧ (L(s) ∨ L(b))) = $\bigcap_{s \in S}$ L(mu{ s, a }∪ mu{ s, b }) = $\bigcap_{s \in S}$ L(s) = L(S). Similarly (L(T) ∨ L(a)) ∧ (L(T) ∨ L(b)) = L(T). Therefore φ is a bijective map and by similar argument we can show φ satisfies the condition: L(S) ≤ L(T) if and only if φ(L(S)) ≤ φ(L(T)). Therefore φ is an order isomorphism.

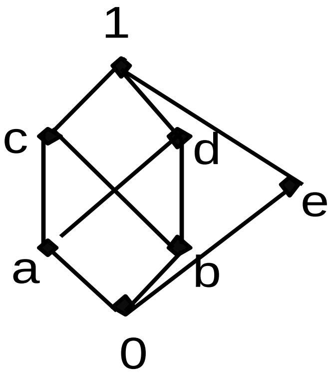
5.7. Theorem Let P be an integral gl-groupoid satisfying the condition (4). If P is complemented then P is distributive and L({ x, y }) = L(xy) for any x, y ∈ P.

Proof: Let a ∈ P and a^1 be a complement of a in P. Clearly a, a^1 are coprime in P. By above theorem, L (P) = [L(0), L(e)] = [L({ a, a^1 }), L(e)] ≅ [L(a), L(e)] × [L(a^1), L(e)] in L (P). Then L(a), L(a^1) are belongs to center of L (P). Therefore L(a) is distributive in L (P), that is a is distributive in P. Let x ∈ P and x^1 be a complement of x in P. By Lemma 4.3, mu{ xa, xa^1 } = { x } and mu{ xa, x^1a } = { a }. Then U(ML{ x, a }) = U(ML(mu{ xa, xa^1 } ∪ mu{ xa, x^1a })) = U({ xa, xa^1 }) V U({ xa, x^1a }) = (U(xa) V U(xa))∩ (U(xa) V U(x^1a))∩ (U(xa^1) V U(xa)) ∩ (U(xa^1) V U(x^1a)) = U(xa). Therefore L({ x, a }) = L(mu(ML{ x, a })) = L(xa).

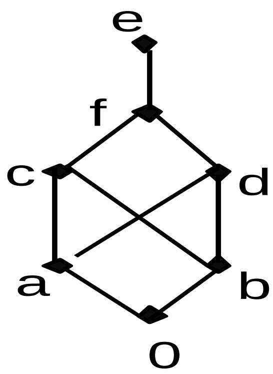
Figures



fig(1)



fig(2)



fig(3)

6. Conclusion

It has been proved that any complemented integral gl-groupoid satisfying a condition is distributive.

Acknowledgements

The author would like to thank Professor M. Krishna Murty for their valuable suggestions and comments during preparation of the paper.

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