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Application of Balancing Numbers in Effectively Solving Generalized Pell's Equation

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Abstract: *Solving a generalized Pell's equation of the form ,* $y^2 - Dx^2 = k$ *basically involves two steps.*

The first step is to find out the primitive solutions for the same, and the second step is to solve related Pell's equation $y^2 - Dx^2 = 1$*; and combine the two solutions. Therefore it is evident that the number of independent primitive solutions determines the number of independent solution sets.*

In our work, we have tried to remove this dependency on the primitive solutions, while solving a particular class of generalized Pell's equation i.e. equation of the $y^2 - 5a^2x^2 = 4a^2$ *form, where* a *is a constant. We show that the solution set thus obtained is same as the solution set obtained through the classical means.*

Keywords: *Balancing Number, Sequence Balancing Number, Pell's equation, generalized Pell's equation*

MSC: 11A25, 11D25, 11D41

1. Introduction

We have an extensive set of literature relating to finding out independent sets of primitive solutions for a Generalized Pell's equation as evident from brute force search [8],[6], LMM Algorithm, [2],[3],[7].

The rest of the paper is organized as follows. Section 2 provides an introduction on Balancing Numbers, and sets up the tone for the basic problem setup. Section 3 provides details on an existing Classical Approach to solve a specialized version of the problem. Section 4 introduces our proposed approach, and proves that the solution set obtained via our proposed approach is exactly the same compared to the Classical approaches.

Section 5 contains the main result of this paper. It shows the equivalence of solutions in a generalized setting. Section 6 provides the required conclusion.

2. Balancing Numbers

A positive integer *n* is called a *balancing number* [1] if
$1 + 2 + \cdots \ldots \ldots + n - 1 \;=\; (n+1) + (n+2) + \;\ldots \ldots \ldots + (n+r)$, where *r* is called *balancer*.

For a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, Panda [4] defined a number a_m of this sequence a *sequence balancing number* if

$$a_1 + a_2 + \dots\dots\dots + a_{m-1} = a_{m+1} + a_{m+2} + \dots\dots\dots + a_{m+r} \tag{1}$$

for some natural number *r*.

Define, $S_m = a_1 + a_2 + \dots\dots\dots + a_{m-1} + a_m$

Now, equation (1) can be rewritten as

$$S_{m-1} = S_{m+r} - S_m \tag{2}$$

Please note that, here we are making a small change to the definition of the *Sequence Balancing Numbers*. Given a particular sequence $\{a_m\}$, the original definition for *Sequence Balancing Numbers* as defined in (1) asks us to find out the integer values of *m* and *r* that satisfies the given equation. But we are interested in solving for (2), this effectively means that we try to find a sequence that satisfies (2) for all values of *m*. Note that we are simply tweaking the definition of *Sequence Balancing Numbers*. The equation (2) is a *homogeneous linear Recurrence Relation*. Writing (2) in terms of *characteristic polynomials*, we get $z^{m-1} + z^m = z^{m+r}$

Simplifying it yields,

$$1 + z = z^{r+1} \tag{3}$$

For simplicity, let us take $r = 1$. So the objective has been remodified to find out a sequence of numbers that satisfies (2), for all values of *m*, assuming *r* equals 1. Solving (3), the expression for S_m turns out to be

$$S_m = A\left(\frac{1+\sqrt{5}}{2}\right)^m + B\left(\frac{1-\sqrt{5}}{2}\right)^m \tag{4}$$

where A and B are constants. Since A and B are arbitrary constants, let us take $A = B$. Hence, the expression (4) can be rewritten as,

$$S_m = A\left[\left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m\right] \tag{5}$$

Therefore, the mth term of the sequence a_m is obtained as

$$a_m = S_m - S_{m-1}$$

On solving a little for a_m we get

$$a_m = \frac{A}{2^{m-2}}\left[(1+\sqrt{5})^{m-2} + (1-\sqrt{5})^{m-2}\right];\ m \geq 2$$

Since, we have $S_2 = 3A$ and $a_2 = 2A$, therefore we can derive that $a_1 = A$. Hence, the complete solution for a_m can be written as,

$$a_m = \begin{cases} \frac{A}{2^{m-2}}\left[(1+\sqrt{5})^{m-2} + (1-\sqrt{5})^{m-2}\right] & if\ m \geq 2 \\ A & if\ m = 1 \end{cases} \tag{6}$$

In the next section, we use this definition and formulate the Pell's equation, that we are interested in solving. We also provide the solutions for the same, via an existing classical approach.

3. The Classical Approach

The above equation for $m \geq 2$ looks similar to the mth term of the *Fibonacci sequence* denoted by F_m where,

$$F_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right]$$

Now some simple algebraic manipulations between a_m and F_m yields the following relation,

$$a_m^2 - 5A^2F_{m-2}^2 = 4A^2, \quad if\ m\ is\ even$$
$$a_m^2 - 5A^2F_{m-2}^2 = -4A^2, \quad if\ m\ is\ odd \tag{7}$$

For the time being, let us assume that the constant $A = 1$. Later on we will see what happens if the constant is not chosen as 1.

Without loss of generality, let us consider the equation $a_m^2 - 5F_{m-2}^2 = 4$. This equation looks like a generalized Pell's equation, $y^2 - 5x^2 = 4$, with the solution $(x, y) = (F_{m-2}, a_m), m \geq 2, m\ is\ even.$

Also, since we have a well-established method for solving generalized Pell's equation, using that method to solve $y^2 - 5x^2 = 4$ yields the following three sets of recurrence relations,

$$X_n = \pm \frac{(9-4\sqrt{5})^n - (9+4\sqrt{5})^n}{\sqrt{5}}$$
$$Y_n = \pm(9-4\sqrt{5})^n - (9+4\sqrt{5})^n \tag{8}$$

$$X_n = \pm \frac{1}{10}\left[(5+3\sqrt{5})(9-4\sqrt{5})^n + (5-3\sqrt{5})(9+4\sqrt{5})^n\right]$$
$$Y_n = \pm \frac{1}{2}\left[(3+\sqrt{5})(9-4\sqrt{5})^n + (3-\sqrt{5})(9+4\sqrt{5})^n\right] \tag{9}$$

$$X_n = \pm \frac{1}{10}\left[(5+3\sqrt{5})(9+4\sqrt{5})^n + (5-3\sqrt{5})(9-4\sqrt{5})^n\right]$$
$$Y_n = \pm \frac{1}{2}\left[(3+\sqrt{5})(9+4\sqrt{5})^n + (3-\sqrt{5})(9-4\sqrt{5})^n\right] \tag{10}$$

So, at this point, we have the solutions via the Classical Approach. Again equation (7), also states that $(x, y) = (F_{m-2}, a_m), m \geq 2$ satisfy the above equation $y^2 - 5x^2 = 4$ for only even values of m. We shall use this property in the next section, to develop our proposed method, and prove equivalence of results.

4. Comparing Classical Approach with our Proposed Method

Now, rewriting $(x, y) = (F_{m-2}, a_m), m \geq 2$ within the indices m belonging to the set of integers, we have,

$$x_n = \pm \frac{1}{4^{n-1}\sqrt{5}}\left[(6-2\sqrt{5})^{n-1} - (6+2\sqrt{5})^{n-1}\right]$$
$$y_n = \pm \frac{1}{4^{n-1}}\left[(6-2\sqrt{5})^{n-1} + (6+2\sqrt{5})^{n-1}\right], n \in \mathbb{Z} \tag{11}$$

We now tend to verify the solutions obtained in (11) against the standard sets of solutions obtained in (8),(9),(10). Infact, it can be observed that,

$$Y_k = \begin{cases} y_{3k+1}\ from\ (8)\ and\ (11) \\ y_{3k}\ \ \ \ from\ (9)\ and\ (11) \\ y_{3k+2}\ from\ (10)\ and\ (11) \end{cases}$$

$$X_k = \begin{cases} x_{3k+1}\ from\ (8)\ and\ (11) \\ x_{3k}\ \ \ \ from\ (9)\ and\ (11) \\ x_{3k+2}\ from\ (10)\ and\ (11) \end{cases}$$

Hence, the set of x, y-values obtained from the relations (8), (9), (10) is equal to the set of x, y-values obtained in (11).

So, instead of having three recurrence relations as solutions for the above Pell's equation, we now have one new relation, which is equivalent to the three (older) relations.

5. The Generalized Case

5.1. Using the Classical Approach

Let us consider the generalized case now, i.e. when the constant$A \neq 1$. So the equation that is under consideration is,

$$y^2 - 5a^2x^2 = 4a^2 \tag{12}$$

We need to find out how the solution obtained by the standard method relates to the solution set (F_{m-2}, a_m) as evident from (7). First of all, let us use the standard method to solve the above equation.

Consider the related Pell's equation $y^2 - 5a^2x^2 = 1$. Rewriting$ax = t$, this becomes a standard Pell's equation, whose solution is,

$$y = \pm\frac{1}{2}[-(9-4\sqrt{5})^n - (9+4\sqrt{5})^n]$$

$$t = \pm\frac{1}{2\sqrt{5}}[(9-4\sqrt{5})^n - (9+4\sqrt{5})^n], n \in \mathbb{Z}$$

As $ax = t$, given a fixed value of a, we need to find $n \in \mathbb{Z}$, such that a divides

$$\pm\frac{1}{2\sqrt{5}}[(9-4\sqrt{5})^n - (9+4\sqrt{5})^n] \tag{13}$$

Therefore the minimal solution of the Pell's equation $y^2 - 5a^2x^2 = 1$ turns out to be(x_0, y_0), where

$$x_0 = \frac{1}{2a\sqrt{5}}[(9+4\sqrt{5})^n - (9-4\sqrt{5})^n]$$

$$y_0 = \frac{1}{2}[(9+4\sqrt{5})^n + (9-4\sqrt{5})^n] \tag{14}$$

And the complete solution is given by (x_n, y_n) where,

$$x_n = \pm\frac{1}{2a\sqrt{5}}[(y_0+\sqrt{5}ax_0)^n - (y_0-\sqrt{5}ax_0)^n]$$

$$y_n = \pm\frac{1}{2}[(y_0+\sqrt{5}ax_0)^n + (y_0+\sqrt{5}ax_0)^n], n \in \mathbb{Z} \tag{15}$$

Let us assume that (x, y) is one of the primitive solutions for$y^2 - 5a^2x^2 = 4a^2$. Hence, the complete solution of $y^2 - 5a^2x^2 = 4a^2$, corresponding to the primitive solution (x, y) can be written as (X_n, Y_n), where

$X_n = yx_n \pm xy_n$ and $Y_n = yy_n \pm 5a^2xx_n$

Substituting (x_n, y_n) from (15) into the above equations, we have

$$X_n = \left(\pm\frac{y}{2a\sqrt{5}} \pm \frac{x}{2}\right)(y_0+\sqrt{5}ax_0)^n - \left(\pm\frac{y}{2a\sqrt{5}} \mp \frac{x}{2}\right)(y_0-\sqrt{5}ax_0)^n$$

$$Y_n = \left(\pm\frac{y}{2} \pm \frac{\sqrt{5}ax}{2}\right)(y_0+\sqrt{5}ax_0)^n + \left(\pm\frac{y}{2} \mp \frac{\sqrt{5}ax}{2}\right)(y_0-\sqrt{5}ax_0)^n \tag{16}$$

Now, from (14), we get the following two relations,

$$y_0 + \sqrt{5}ax_0 = (9+4\sqrt{5})^n \text{ and } y_0 - \sqrt{5}ax_0 = (9-4\sqrt{5})^n \tag{17}$$

Therefore, from (16) and (17) we get two distinct set of values for (X_n, Y_n), which are as follows,

$$X_n = \pm\frac{1}{2a\sqrt{5}}\left[(y+\sqrt{5}ax)(9+4\sqrt{5})^{an} - (y-\sqrt{5}ax)(9-4\sqrt{5})^{an}\right]$$

$$Y_n = \pm\frac{1}{2}\left[(y+\sqrt{5}ax)(9+4\sqrt{5})^{an} + (y-\sqrt{5}ax)(9-4\sqrt{5})^{an}\right] \tag{18}$$

And

$$\hat{X}_n = \pm\frac{1}{2a\sqrt{5}}\left[(y-\sqrt{5}ax)(9+4\sqrt{5})^{an} - (y+\sqrt{5}ax)(9-4\sqrt{5})^{an}\right]$$

$$\hat{Y}_n = \pm\frac{1}{2}\left[(y-\sqrt{5}ax)(9+4\sqrt{5})^{an} + (y-\sqrt{5}ax)(9-4\sqrt{5})^{an}\right] \tag{19}$$

We now claim that $(2ax_n, 2ay_n)$ where (x_n, y_n) as defined in (15) is also a solution to $y^2 - 5a^2x^2 = 4a^2$, since

$$(2ay_n)^2 - 5a^2(2ax_n)^2 = 4a^2(y_n^2 - 5a^2x_n^2) = 4a^2$$

as (x_n, y_n) is a solution for the associated Pell's equation $y^2 - 5a^2x^2 = 1$. So corresponding to one primitive solution(x, y), we have now two set of solutions for the equation, $y^2 - 5a^2x^2 = 4a^2$, and one common solution $(2ax_n, 2ay_n)$ across all primitive solutions.

5.2. Using our Proposed Method for Generalized Case

The expressions mentioned in (11) solve the equation $y^2 - 5a^2x^2 = 4a^2$ for $a = 1$. Rewriting those for the generalized case, we have that the expressions,

$$x_n = \pm\frac{1}{4^{n-1}\sqrt{5}}\left[(6-2\sqrt{5})^{n-1} - (6+2\sqrt{5})^{n-1}\right]$$

$$y_n = \pm\frac{a}{4^{n-1}}\left[(6-2\sqrt{5})^{n-1} + (6+2\sqrt{5})^{n-1}\right], n \in \mathbb{Z} \tag{20}$$

solves the generalized equation $y^2 - 5a^2x^2 = 4a^2$ for any generalized value of a.

Let us assume for a moment that if (x, y) is a primitive solution for the equation$y^2 - 5a^2x^2 = 4a^2$, then we must have $y + \sqrt{5}ax = 2a(\frac{3+\sqrt{5}}{2})^{t-1}$ for some $0 < t < 3a - 1, t \in \mathbb{Z}$. Therefore, under this assumption we can prove, $y - \sqrt{5}ax = 2a(\frac{3-\sqrt{5}}{2})^{t-1}$ as with these definitions, we have $y^2 - 5a^2x^2 = 4a^2$. If these assumptions were true, let us see the repercussions first, and then we shall go ahead and prove it. Now replacing n by $3a + t$, with $0 < t < 3a - 1, t \neq 1, t \in \mathbb{Z}$ in (20), we get,

$$x_{3ak+t} = \pm\frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{t-1}(9+4\sqrt{5})^{ak} - \left(\frac{3-\sqrt{5}}{2}\right)^{t-1}(9-4\sqrt{5})^{ak}\right]$$

$$y_{3ak+t} = \pm a\left[\left(\frac{3-\sqrt{5}}{2}\right)^{t-1}(9-4\sqrt{5})^{ak} - \left(\frac{3+\sqrt{5}}{2}\right)^{t-1}(9+4\sqrt{5})^{ak}\right] \tag{21}$$

Now, under our assumption $y + \sqrt{5}ax = 2a(\frac{3+\sqrt{5}}{2})^{t-1}$ and $y - \sqrt{5}ax = 2a(\frac{3-\sqrt{5}}{2})^{t-1}$ for some $0 < t < 3a - 1, t \neq 1, t \in \mathbb{Z}$, equation (21) becomes,

$$x_{3ak+t} = \pm\frac{1}{2a\sqrt{5}}\left[(y+\sqrt{5}ax)(9+4\sqrt{5})^{ak} - (y-\sqrt{5}ax)(9-4\sqrt{5})^{ak}\right]$$

$$y_{3ak+t} = \pm\frac{1}{2}[(y+\sqrt{5}ax)(9+4\sqrt{5})^{ak} + (y-\sqrt{5}ax)(9-4\sqrt{5})^{ak}] \tag{22}$$

which is exactly the same as equation (18). Replacing t by $2-t$ in equation (21) and applying our assumption $y+\sqrt{5}ax = 2a(\frac{3+\sqrt{5}}{2})^{t-1}$ and $y-\sqrt{5}ax = 2a(\frac{3-\sqrt{5}}{2})^{t-1}$ for some $0 < t < 3a-1, t \neq 1,\ t \in \mathbb{Z}$, we have,

$$x_{3ak-t+2} = \pm\frac{1}{2a\sqrt{5}}[(y+\sqrt{5}ax)(9-4\sqrt{5})^{ak} - (y-\sqrt{5}ax)(9+4\sqrt{5})^{ak}]$$

$$y_{3ak-t+2} = \pm\frac{1}{2}[(y+\sqrt{5}ax)(9-4\sqrt{5})^{ak} + (y-\sqrt{5}ax)(9+4\sqrt{5})^{ak}] \tag{23}$$

which is exactly same as equation (19). Summarizing the entire thing, we have that for every primitive solution (x,y) of the equation $y^2 - 5a^2x^2 = 4a^2$, we have got two set of solutions, one being (X_n, Y_n) as mentioned in (18) and the other being $(\hat{X}_n, \hat{Y}_n)$ as mentioned in (19). Under our assumption $y+\sqrt{5}ax = 2a\left(\frac{3+\sqrt{5}}{2}\right)^{t-1}$ and $y-\sqrt{5}ax = 2a\left(\frac{3-\sqrt{5}}{2}\right)^{t-1}$ for some $0 < t < 3a-1, t \neq 1, t \in \mathbb{Z}$, we have shown that $(X_n, Y_n) = (x_{3an+t}, y_{3an+t})$ as evident from (22) and $(\hat{X}_n, \hat{Y}_n) = (x_{3an-t+2}, y_{3an-t+2})$ as evident from (23). Also, if $t = 1$, substitute this value in (21), we can see that $(x_{3an+1}, y_{3an+1}) = (2ax_n, 2ay_n)$ where (x_n, y_n) on the *RHS* of this equation is the generic solution of the associated Pell's equation $y^2 - 5a^2x^2 = 1$.

Or in other words, under the assumption, for an arbitrarily chosen primitive solution, the set of solutions obtained via the standard method is a subset of the set of solutions obtained using the *Balancing Numbers* concept.

Now, the only thing left to be done is to show that the assumption in fact holds true, *i.e.* if (x,y) is a primitive solution of $y^2 - 5a^2x^2 = 4a^2$, then there exists a $t, 2 \leq t \leq 3a-1$, such that

$$y+\sqrt{5}ax = 2a\left(\frac{3+\sqrt{5}}{2}\right)^{t-1}$$

holds true.

Let us prove the above implication. We make the following claim now.

Claim: If (x,y) is a primitive solution of $y^2 - 5a^2x^2 = 4a^2$, then a divides y i.e. $y = at$ for some $t \in \mathbb{Z}$.

Proof: The proof is quite intuitive, i.e. from the original expression, we get $y = a\sqrt{4+5x^2}$. Since $x, y \in \mathbb{Z}$ and x, y satisfy the equation $y = a\sqrt{4+5x^2}$, therefore $4+5x^2$ must be a perfect square. Hence, $y = at$, where $t = \sqrt{4+5x^2}, t \in \mathbb{Z}$, or in other words $a|y$.

So, now the primitive solutions for $y^2 - 5a^2x^2 = 4a^2$, are of the form (p, aq) for some $p, q \in \mathbb{Z}$. Substituting in the equation, we get $a^2q^2 - 5a^2p^2 = 4a^2 \Rightarrow q^2 - 5p^2 = 4$ as $\geq 1, a \in \mathbb{Z}$. Or in other words, (p,q) satisfy the equation $y^2 - 5x^2 = 4$, and we have already seen earlier that $(F_{m-2}, a_m), m \in \mathbb{Z}$ satisfies this equation.

Hence, the primitive solutions for $y^2 - 5a^2x^2 = 4a^2$, are of the form $(F_{m-2}, aa_m), m \in \mathbb{Z}$. All that remains, is the range of values that m can take, so that the resulting pair (F_{m-2}, aa_m) still remains as the primitive solution.

Using, the theory of bounds on primitive solutions for generalized Pell's equation [1], [4], we obtain the following relation, i.e. if (x, y) is a primitive solution to the equation, $y^2 - 5a^2x^2 = 4a^2$, then

$$y \leq a\left[\left(\sqrt{5}+2\right)^n + \left(\sqrt{5}-2\right)^n\right] \text{ and } y > 0 \tag{24}$$

Therefore, for a given fixed a, and the fact stated earlier that the yth component of the primitive solution is of the form aa_m, we use the expression stated in (24) to find out the bounds for m, i.e. the number of possible primitive solutions.

Writing out the expression for y, we have

$$y_n = \pm\frac{a}{4^{n-1}}\left[(6-2\sqrt{5})^{n-1} + (6+2\sqrt{5})^{n-1}\right], n \in \mathbb{Z} \tag{25}$$

So, substituting (25) in (24), for a fixed a we get the following relation, $1 \leq n < 1 + 1.5a$.

Summarizing the contents of the proof till now, we have that (F_{m-2}, aa_m) forms the primitive solution for $y^2 - 5a^2x^2 = 4a^2$, for a fixed a, where F_m is the mth Fibonacci number and a_m equals y_n as defined in (11). Also, the number of such primitive solutions is given by the inequality $1 \leq n < 1 + 1.5a$.

Now, consider the expression $aa_m + \sqrt{5}aF_{m-2}$. Substitute values from (11) to get,

$$aa_m + \sqrt{5}aF_{m-2} = 2a\left(\frac{3+\sqrt{5}}{2}\right)^{m-1} \tag{26}$$

So, since the primitive solution (x, y) of $y^2 - 5a^2x^2 = 4a^2$ are of the form (F_{m-2}, aa_m), from (26) we get that $y + \sqrt{5}ax = 2a\left(\frac{3+\sqrt{5}}{2}\right)^{t-1}$ for $1 \leq t \leq 3a - 1$, where $t = m$. Since, we have earlier proved that $1 \leq m < 1 + 1.5a$, expression (26) effectively proves that, given a fixed a, for every primitive solution of the equation $y^2 - 5a^2x^2 = 4a^2$, we can find a t, $1 \leq t \leq 3a - 1$ such that $y + \sqrt{5}ax = 2a\left(\frac{3+\sqrt{5}}{2}\right)^{t-1}$. Hence, the assumption is proved.

Conversely, in order to show that the set of solutions obtained via the *Balancing Numbers* concept is also a subset of the solutions obtained via the standard method, all we need to show is that the implication

$$2a\left(\frac{3+\sqrt{5}}{2}\right)^{t-1} = p + \sqrt{5}aq \Longrightarrow p^2 - 5a^2q^2 = 4a^2$$

hold for all values of t, $2 \leq t \leq 3a - 1$. If it does, then equation (21) can be written in the form of equation (18) for all values of t, $2 \leq t \leq 3a - 1$. And earlier we have already proved that the relations obtained in (18) belong to the set of solutions obtained via the standard method. Hence, if the above implication is proven true, then the set of solutions obtained via the *Balancing Numbers* concept will also be a subset of the solutions obtained via the standard method. Now, the above implication can be proved via *Mathematical Induction*. Set $t = 2$, we have

$$2a\left(\frac{3+\sqrt{5}}{2}\right)^{t-1} = 3a + \sqrt{5}a \ \ and \ \ (3a)^2 - 5a^2.1 = 4a^2$$

Let the above implication be true for any $t = k$ *i.e.* $2 \leq t = k \leq 3a - 1$, we have

$$2a\left(\frac{3+\sqrt{5}}{2}\right)^{k-1} = p+\sqrt{5}aq \ \ and \ \ p^2-5a^2q^2=4a^2$$

Let us assume that k is *odd*, $k-1$ is *even*. So, we have that both p,q are *odd*. Now,

$$2a\left(\frac{3+\sqrt{5}}{2}\right)^{k} = (p+\sqrt{5}aq)\left(\frac{3+\sqrt{5}}{2}\right) = \left(\frac{3p+5aq}{2}+\sqrt{5}\frac{p+3aq}{2}\right)$$

As p,q are *odd*, $\frac{3p+5aq}{2}$ and $\frac{p+3aq}{2}$ are integers, and

$$\left(\frac{3p+5aq}{2}\right)^2 - 5a^2\left(\frac{p+3aq}{2}\right) = 4a^2$$

A similar case can be proven when k is *even*. This concludes the proof of the above implication. This concludes the proof of the fact that the set of solutions obtained via the classical approach is a subset of the set of solutions obtained using the *Balancing Numbers* theory.

6. Conclusion

We have successfully shown that for the equation $y^2-5a^2x^2=4a^2$, the solution set obtained via the standard method equals the solution set obtained by the *Balancing Number* concept. The standard method, requires to solve for some certain number of primitive solutions, and then obtain the complete set, whereas the *Balancing Number* concept simply gives a single recurrence relation that is equivalent to the complete solution set obtained by the standard method, a fact that was introduced and proved in this paper.

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