On $|N, q_n, r_n|$- Summability of Jacobi Series

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Abstract: In this paper we have established a theorem on $|N, q_n, r_n|$-summability of Jacobi series, which gives some new interesting results and generalizes some previous known results.

Keywords: $|N, q_n, r_n|$-summability method and Jacobi series.

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1. INTRODUCTION

Let $f(x)$ be a function defined on the interval $-1 \leq x \leq 1$ such that the integral

$$
\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x) \,dx
$$

(1.1)

Is exist in the sense of Lebesgue for $\alpha > -1$ and $\beta > -1$. The Jacobi series corresponding to the function $f(x)$ is given by

$$
f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha, \beta)}(x)
$$

(1.2)

Where

$$
a_n = \frac{(2n + \alpha + \beta + 1)\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha + \beta + 1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)} \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) \,dx
$$

If

$$
b_n = \frac{(2n + \alpha + \beta + 1)\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha + \beta + 1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}
$$

(1.3)

Then

$$
a_n = b_n \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) \,dx
$$

(1.4)

and $P_n^{(\alpha, \beta)}(x)$ are the Jacobi polynomials defined by the generating function

$$
2^{\alpha + \beta} (1 - 2xt + t^2)^{-1/2} \left[1 - t + (1 - 2xt + t^2)^{1/2}\right]^{-\alpha} \times \left[1 + t + (1 - 2xt + t^2)^{1/2}\right]^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x)t^n
$$

(1.5)

Let us write
\[ F(\phi) = \{ f(\cos \phi) - A \}/2 \left( (\sin \phi / 2)^{2^{n+1}} \right) \]

where \( A \) being a constant.

Let \( \{s_n\} \) be the sequence of partial sums of an infinite series \( \Sigma a_n \). Let \( \{r_n\} \) and \( \{q_n\} \) be any two sequences of positive real constants with \( R_n \) and \( Q_n \) as their \( n \)-th partial sums respectively and let

\[
(q \ast r)_n = \sum_{k=0}^{n} q_{n-k}r_k = \sum_{k=0}^{n} q_k r_{n-k}
\]
tends to infinity as \( n \to \infty \). (1.6)

If the sequence to sequence transformation is defined by (Borwein [1])

\[
t^q_s r_s = \frac{1}{(q \ast r)_n} \sum_{k=0}^{n} q_{k}r_{s-k}
\]

(1.7)

If

\[
t^q_s r_s \to s \quad \text{as} \quad n \to \infty
\]

then the sequence of partial sums \( \{s_n\} \) or infinite series \( \Sigma a_n \) is said to be summable \( |N, q_n, r_n| \) to \( s \).

2. KNOWN RESULTS

Dealing with Nörlund summability of Jacobi series Pandey [7] has established the following theorem.

**Theorem 2.1**

Let \( \alpha > - \frac{1}{2}, \quad \beta - \alpha > -1, \quad \beta + \alpha \geq -1 \). Suppose that

\[
\sum_{k=2}^{n} \frac{Q_n}{k^{n+\alpha+(1/2)} \log k} = O \left( \frac{Q_n}{n^{\alpha+(1/2)}} \right), \text{ as } n \to \infty.
\]

(2.1)

Also suppose that

\[
\int_{1-\epsilon}^{1} | f(x) - A | \, dx = O \left( \frac{t}{\log (\frac{1}{\epsilon})} \right), \quad t \to 0
\]

(2.2)

and that the antipole condition

\[
\int_{-\epsilon}^{b} (1 + x)^{(\beta - \alpha - 1)/2} | f(x) | \, dx < \infty
\]

(2.3)

is satisfied, where \( b \) is fixed then the series (1.2) is summable \( |N, q_n| \) at the point \( x = +1 \) to the sum \( A \).

3. MAIN RESULTS

The object of this paper is to generalize the Theorem 2.1 to a more general class on \( |N, q_n, r_n| \) - summability of the Jacobi series.

**Theorem 3.1**

Let \( (N, q_n, r_n) \) be a summability method defined by a non-negative real constants sequences \( \{q_n\} \)

and \( \{r_n\} \) and let \( \alpha > - \frac{1}{2}, \beta - \alpha > 1, \beta + \alpha \geq -1 \) such that
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\[
\sum_{k=2}^{n} \frac{(q^* r)_k}{k^{n+(1/2)} \log k} = O \left( \frac{(q^* r)_n}{n^{a+(1/2)}} \right) \quad \text{as } n \to \infty
\]  

(3.1)

Also suppose that

\[
\int_{1/2}^{1} |f(u) - A| \, du = O \left( \frac{t}{\log (1/t)} \right) \text{, as } t \to O
\]  

(3.2)

and the antipole condition

\[
\int_{-1}^{b} (1 + x)^{(d - d - 1)/2} |f(x)| \, dx < \infty
\]  

(3.3)

are satisfied where \(b\) is fixed then the series \((1.2)\) is summable \(|N, q_n, r_n| \) at \(x = +1\) to the sum \(A\).

4. LEMMAS

We have required the following lemmas to prove the theorem:

**Lemma 4.1**

Szego [10] for \(\alpha > -1, \beta > -1\)

\[
P^{(a, b)}_n (\cos \phi) = \begin{cases} 
O(n^a), & \text{when } 0 \leq \phi \leq 1/n \\
O(n^\beta), & \text{when } \pi - \frac{1}{n} \leq \phi \leq \pi \\
\frac{1}{(a+\beta+1)^{1/2}} \left( \sin \frac{\phi}{2} \right)^{(2a+1)/2} \left( \cos \frac{\phi}{2} \right)^{(2\beta+1)/2} \left( \cos \left( \frac{2n + a + \beta + 1}{2} \right) \phi - (2\alpha + 1) \frac{\pi}{4} \right) + O(1), & \text{when } \frac{1}{n} \leq \phi \leq \pi - \frac{1}{n}
\end{cases}
\]

**Lemma 4.2**

Let \(\alpha > -\frac{1}{2}, \beta > -1\) and also let

\[
N_n(\phi) = \frac{1}{(q^* r)_n} (2)^{a+\beta+1} \sum_{k=0}^{n} q_k r_{n-k} \lambda_{n-k} P^{(a+1, \beta)}_{n-k} (\cos \phi)
\]

Where

\[
\lambda_{n-k} = \frac{2^{-a-\beta-1} \Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)} \sim \frac{2^{-a-\beta-1} n^{a+1}}{\Gamma(\alpha + 1)}
\]

Then

For \(0 \leq \phi \leq \frac{1}{n}\)

\[
|N_n(\phi)| = O(n^{2a+2})
\]  

(4.2)

For \(\frac{1}{n} \leq \phi \leq \pi - \frac{1}{n}\)

\[
|N_n(\phi)| = O \left( \frac{n^{1/2}}{(q^* r)_n} \left( \frac{\sin \phi}{2} \right)^{(2a+1)/2} \left( \cos \phi \right)^{(2\beta+1)/2} \right)
\]  

(4.3)
For
\[ \pi - \frac{1}{n} \leq \phi \leq \pi \]
\[ |N_n(\phi)| = O(n^{\alpha + \beta + 1}) \]  \hspace{1cm} (4.4)

**Proof:**

For \( \alpha > -\frac{1}{2} \) and \( \beta > -1 \) and \( \{q_n\} \) and \( \{r_n\} \) satisfy the conditions of theorem, using Lemma 4.1 for \( 0 \leq \phi \leq \frac{1}{n} \) then condition (4.2) is satisfied. For the estimation of (4.3) we use the Lemma (4.1) and Lemma (4.3) for \( \pi - \frac{1}{n} \leq \phi < \pi \).

For \( \frac{1}{n} \leq \phi \leq \pi - \frac{1}{n} \) we have
\[ N_n(\phi) = O\left(\frac{1}{(q \ast r)^n}\sum_{k=1}^{n-1} q_k r_{n-k} (n-k)^{(2\alpha+1)/2} \left(\sin \frac{\phi}{2}\right)^{(2\alpha+1)/2} \left(\cos \frac{\phi}{2}\right)^{(2\beta+1)/2}\right) \]
\times \left[\cos\{(n-k+\rho)\phi - \gamma\} + \frac{O(1)}{(n-k)\sin \phi}\right]

Since for fixed \( n, (r_{n-k}) \) is non-increasing we can deal with the first term of the right by using the second mean value theorem and apply the result of Lemma (4.3) and the required estimate follows.

**Lemma 4.3:** (Khare [5]) If \( \{q_n\} \) is a non-negative, non-increasing and \( \{r_n\} \) is a non-negative, non-decreasing sequence, then
\[ \sum_{k=0}^{n-1} q_k r_{n-k} (n-k)^{(2\alpha-1)/2} = O\left((q \ast r)_n n^{(2\alpha-1)/2}\right) \]

**Lemma 4.4:** The assumption (3.1) implies that
\[ n^{\alpha + (\frac{1}{2})} = O\{q \ast r\}_n \]  \hspace{1cm} (4.5)
where \( \alpha < \frac{1}{2} \)  \hspace{1cm} (4.6)

**Proof:** The expression on the left of (3.1) is increasing and hence greater than equal to a positive constant. Hence (3.1) implies that, for some positive constant \( c \)
\[ (q \ast r)_n > cn^{(2\alpha+\frac{1}{2})} \]
On substituting this, we see that the expression on the left of (3.1) tends to \( \infty \) as \( n \to \infty \) and (4.5) follows.

Since \( q_n \) and \( r_n \) are positive non increasing, \( (q \ast r)_n = O(n) \) and (4.6) therefore follows by (4.5).

**Lemma 4.5:** (Pandey [7]) condition (3.2) is equivalent to
\[ F(t) = \int_0^t F(\phi) d\phi = O\left(\frac{t^{2\alpha+2}}{\log(1/t)}\right) \text{ as } t \to 0 \]  \hspace{1cm} (4.7)
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Where

$$F(\phi) = \left[ f(\cos \phi) - A \right] \left( \frac{\phi}{2} \right)^{2n+1} \cos \frac{\phi}{2}.$$

**Lemma 4.6:** Let $\beta - \alpha > -1$. The antipole condition

$$\int_{-1}^{1} (1 + x)^{(\beta - \alpha - 1)/2} | f(x) | dx < \infty$$

is equivalent to

$$\int_{-1}^{1} (1 + x)^{(\beta - \alpha - 1)/2} | f(x) - A | dx < \infty$$

Further

$$\int_{a}^{\pi} (\cos \frac{\phi}{2})^{-\beta - 1} | f(\phi) | d\phi < \infty \quad (4.8)$$

5. **Proof of Theorem**

The nth partial sum of the series (1.2), at the point $x = +1$ is given by Obrechkoff [6].

$$S_{n}(1) = 2^{n+\beta+1} \int_{0}^{\pi} (\sin \frac{\phi}{2})^{2n+1} (\cos \frac{\phi}{2})^{2\beta+1} f(\cos \phi) s_{n}(1, \cos \phi) \sin \phi d\phi \quad (5.1)$$

Where $S_{n}(1, \cos \phi)$ denote the nth partial sum of the series

$$\sum_{n=0}^{\infty} \frac{P_{n}(\alpha, \beta)(1) P_{n}(\alpha, \beta)(\cos \phi)}{b_{n}}.$$

Rao [9] has been shown that

$$S_{n}(1, \cos \phi) = \lambda_{n} P_{n}(\alpha + 1, \beta)(\cos \phi)$$

Therefore

$$S_{n}(1) - A = 2^{n+\beta+1} \lambda_{n} \int_{0}^{\pi} (\sin \frac{\phi}{2})^{2n+1} (\cos \frac{\phi}{2})^{2\beta+1} | f(\cos \phi) - A | P_{n}(\alpha + 1, \beta)(\cos \phi) d\phi$$

$$= 2^{n+\beta+1} \lambda_{n} \int_{0}^{\pi} F(\phi) P_{n}(\alpha + 1, \beta)(\cos \phi) dQ \quad (5.2)$$

The $(N, q_n, r_n)$ means of the series (1.2) of the point $x = +1$ is given by

$$t_{n} = \frac{1}{(q^{*} r_{n})} \sum_{k=0}^{n} q_{k} r_{n-k} s_{n-k}(1)$$

$$t_{n} - A = \frac{1}{(q^{*} r_{n})} \sum_{k=0}^{n} q_{k} r_{n-k} \{ s_{n-k}(1) - A \}$$

$$= \frac{1}{(q^{*} r_{n})} \sum_{k=0}^{n} q_{k} r_{n-k} 2^{n+\beta+1} \lambda_{n-k} \int_{0}^{\pi} F(\phi) P_{n-k}(\alpha + 1, \beta)(\cos \phi) d\phi$$

$$= \int_{0}^{\pi} f(\phi) N_{n}(\phi) d\phi \quad (5.3)$$

Where

$$N_{n}(\phi) = \frac{1}{(q^{*} r_{n})} (2)^{n+\beta+1} \sum_{k=0}^{n} q_{k} r_{n-k} \lambda_{n-k} P_{n-k}(\alpha + 1, \beta)(\cos \phi)$$
To prove the theorem we have to show that
\[ I = \int_0^\pi F(\phi) N_n(\phi) d\phi = O(1), \text{ as } n \to \infty \]

\[ I = \left( \int_0^{1/n} + \int_{1/n}^\delta + \int_{\delta}^{\pi/2} + \int_{\pi/2}^\pi \right) F(\phi) N_n(\phi) d\phi \]

\[ = I_1 + I_2 + I_3 + I_4 \text{ (say)} \quad (5.4) \]

Now
\[ |I_1| = \left| \int_0^{1/n} f(\phi) N_n(\phi) d\phi \right| \]

Using (4.1), we have
\[ I_1 = O\left( n^{2\alpha+2} \right) O\left( \frac{R^{2\alpha-1}}{\log n} \right) \]

\[ = O\left( \frac{1}{\log n} \right) \]

\[ = O(1) \text{ as } n \to \infty \quad (5.5) \]

Next
\[ |I_2| = \left| \int_{1/n}^\delta F(\phi) N_n(\phi) d\phi \right| \]

Using (4.2) we have
\[ I_2 = O\left( \int_{1/n}^\delta f(\phi) \left\{ n^{(2\alpha+1)/2} \right( q \ast r \ast_1(\phi) - q \ast r \ast_2(\phi) \right\} \left( \sin \frac{\phi}{2} \right)^{-(2\alpha+1)/2} d\phi \right) \]

\[ + O\left( \int_{1/n}^\delta f(\phi) \left\{ n^{(2\alpha+1)/2} \right( \sin \frac{\phi}{2} \right)^{-(2\alpha+1)/2} d\phi \right) \]

\[ = I_{2,1} + I_{2,2} \quad (5.6) \]

For given any \( c > 0 \), let be chosen so that
\[ |f_1(\phi)| \leq \frac{c\phi^{2\alpha+2}}{\log(1/\phi)}, \text{ for } 0 \leq \phi \leq \delta \]

Then
\[ |I_{2,1}| \leq \frac{M^{(2\alpha+1)/2}}{q \ast r \ast_1} \int_{1/n}^\delta f(\phi) \left( q \ast r \ast_1(\phi) - q \ast r \ast_2(\phi) \right) \phi^{-(2\alpha+1)/2} d\phi \]

Where, we suppose \( M \) is used throughout the paper to denotes a positive constant, which may be different at each occurrence.

\[ |I_{2,1}| = \frac{M^{(2\alpha+1)/2}}{q \ast r \ast_1} \left\{ \left[ F_1(\phi)(q \ast r \ast_1(\phi) - q \ast r \ast_2(\phi)) \right]_{1/n}^\delta - \int_{1/n}^\delta f_1(\phi) d\left( \frac{q \ast r \ast_1(1/\phi)}{\phi^{(2\alpha+1)/2}} \right) \right\} \]

\[ = I_{2,1,1} + I_{2,1,2} \quad (5.7) \]
Now if \( m(\delta) \) denotes a constant depending on \( \delta \), we have for fixed \( \delta \).

\[
I_{2,3,1} = m(\delta) n \frac{(2a+1/2)}{(q*r)_a} + O\left(\frac{1}{\log n}\right)
\]

\[= O(1) \text{ as } n \to \infty \quad (5.8)\]

And

\[
|I_{2,1,1}| \leq m_n \frac{n(2a+1/2)}{(q*r)_a} \left[ \int_{\log \log (\frac{1}{\delta})}^{\phi(2a+3/2)} d \left( (q*r)_{(2a+3/2)} \right) \right]
\]

\[
\leq m_n \frac{n(2a+1/2)}{(q*r)_a} \left[ \int_{\phi(2a+3/2)}^{\phi(2a+1/2)} d \left( (q*r)_{(2a+1/2)} \right) \right]
\]

\[
\leq m_n \frac{n(2a+1/2)}{(q*r)_a} \left[ \int_{\phi(2a+1/2)}^{\phi(2a+3/2)} d \left( (q*r)_{(2a+3/2)} \right) \right]
\]

\[
= m_n \frac{n(2a+1/2)}{(q*r)_a} \left[ J + ((2a+3/2)k) \right] \quad (say)
\]

Since \((q*r)_{(x)}\) has a jump of \((q*r)_x\) at \(x = k\) therefore

\[
J = \sum_{k=a}^{n} (q*r)_k \frac{1}{\log k}
\]

Where \(a\) is a fixed positive integer

But, since \((q*r)_k\) is non-negative, non-increasing, \((k+1)(q*r)_k \leq O(q*r)_k\) so

\[
J = O\left( \sum_{k=a}^{n} \frac{(q*r)_k}{\log k} \right)
\]

Also

\[
k \leq \sum_{k=a-1}^{n} (q*r)_k \int_{x^{-1/2}}^{1} \frac{d}{\log x}
\]

\[
= O\left( \sum_{k=a-1}^{n} \frac{(q*r)_k}{\log k} \right)
\]

By (3.1) we have

\[
|I_{2,1,1}| \leq m_n \quad (5.9)
\]

Again

\[
|I_{2,2}| \leq m_n \frac{1}{(2a-1/2)} \int_{\log \log (\frac{1}{\delta})}^{\phi(2a-5/2)} d \phi
\]

\[
= n \frac{1}{(2a-1/2)} \left[ m[F_1(\phi)\phi(2a-5/2)]_{1/n}^{\delta} + m \int_{1/n}^{\delta} F_1(\phi)\phi(2a-7/2) d \phi \right]
\]
Now

\[ I_{2,2,1} = m \left( \delta \right) n^{(2a-1)/2} + O \left( \frac{1}{\log n} \right) \]

\[ = O(1) \text{ as } n \to \infty \] (5.10)

Also

\[ |I_{2,2,2}| \leq m_n n^{(2a-1)/2} \int_{1/n}^{\delta} \frac{\phi^{(2a-3)/2}}{\log(1/\phi)} d\phi \]

\[ \leq m_n n^{(2a-1)/2} \int_{1/n}^{\delta} \frac{x^{-(2a-3)/2}}{\log x} dx \]

\[ = m_n \text{ because } \alpha < \frac{1}{2} \] (5.11)

Hence

\[ I_2 = O(1) \text{ as } n \to \infty \] (5.12)

Next

\[ I_3 = O \left( \frac{n^{(2a+1)/2}}{(q*r)_n} \int_{x_{1/n}}^{\delta} \cos \left( \frac{\phi}{2} \right)^{-2\beta -1/2} |F(\phi)| d\phi \right) + O \left( n^{(2a-1)/2} \int_{\phi_{(1/n)}}^{\phi_{1/n}} \cos \frac{1}{2} \phi^{1/2 - \beta -1/2} |F(\phi)| d\phi \right) \]

\[ = I_{3,1} + I_{3,2} \text{ (say)} \]

\[ \alpha \geq -\frac{1}{2} \text{, so that} \]

\[ -\beta - \frac{1}{2} \geq -\beta - \alpha - 1 \]

So that (4.8) Implies that

\[ \int_{\phi_{1/n}}^{\phi_{1/n}} \cos \frac{1}{2} \phi^{1/2 - \beta -1/2} |F(\phi)| d\phi < \infty \]

\[ I_{3,1} = O \left( \frac{n^{(2a+1)/2}}{(q*r)_n} \right) \]

Hence

\[ = O(1) \text{ as } n \to \infty \] (5.13)

Now it is follows from (4.7) that, given any \( \varepsilon > 0 \), we can choose \( \eta > 0 \), so that

\[ \int_{x-\eta}^{x} (\cos(\phi/2))^{-n-\beta-1} |F(\phi)| d\phi < \varepsilon \] (5.14)

Thus, supposing that \( n > \frac{1}{\eta} \), we have

\[ n^{(2a-1)/2} \int_{x-\eta}^{x} (\cos(\phi/2))^{-2\beta -1/2} |F(\phi)| d\phi \]
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$$\leq n^{(2a-1)/2} \left[ \cos \frac{1}{2} (\pi - \frac{1}{n}) \right]^{(2a-1)/2} \int_{\pi - \eta}^{\pi} (\cos \frac{1}{2} \phi)^{-\alpha - \beta - 1} \left| F(\phi) \right| d\phi$$

$$\leq 2 \epsilon$$

by (5.16), provided that $n$ is sufficiently large. But, once $\eta$ has been fixed.

$$n^{(2a-1)/2} \int_{\pi - \eta}^{\pi} (\cos \frac{1}{2} \phi)^{-\alpha - \beta - 1} \left| F(\phi) \right| d\phi$$

Is just a constant, and hence can be made $< \epsilon$, by choosing $n$ sufficiently large. Hence

$$I_3 = O(1) \quad (5.17)$$

Finally, since $\alpha + \beta + 1 > 0$

$$I_4 = O \left( n^{\alpha + \beta + 1} \int_{\pi - \eta}^{\pi} \left| F(\phi) \right| d\phi \right)$$

$$= O \left( \int_{\pi - \eta}^{\pi} (\cos \frac{1}{2} \phi)^{-\alpha - \beta - 1} \left| F(\phi) \right| d\phi \right)$$

$$= O(1) \quad (5.18)$$

Using (5.5), (5.13), (5.17) and (5.18) we have

$$I = O(1)$$

This completes the proof of the theorem.

6. CONCLUSIONS

This theorem has more general result rather than the result of B.N. Pandey [7] that will enrich the literature on Jacobi summability theory.

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REFERENCES

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