Some Curious Polynomial Expressions of the Trigonometric Functions

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Abstract: It is noted that \( \cos (4n+1)x \) and \( \cos (4n-1)x \) are polynomials of degree \( 4n+1 \) and \( 4n-1 \) respectively in \( \cos x \) only. It is worth noting that \( \sin (4n+1)x \) is the same \( (4n+1)^{th} \) degree polynomial as \( \cos (4n+1)x \) with \( x \) replaced by \( \sin x \). In contrast to this \( \sin (4n-1)x \) is the same \( (4n-1)^{th} \) degree polynomial in \( \sin x \) with sign reversed. i.e. if \( \cos (4n+1)x = f_{4n+1}(\cos x) \) then \( \sin (4n+1)x = f_{4n+1}(\sin x) \). If \( \cos (4n-1)x = g_{4n-1}(\cos x) \) then \( \sin (4n-1)x = -g_{4n-1}(\sin x) \).

The results are proved in two ways: Employing De moivre’s and Binomial theorems and by using the properties of trigonometric ratios of complimentary angles.

Keywords: Trigonometric polynomials, \( \sin (nx) \), \( \cos (nx) \).

INTRODUCTION

For ‘n’ a positive integer, \( \cos (nx) \) and \( \sin (nx) \) can be expressed as polynomials in \( \sin x \) and \( \cos x \) using Demoivre’s and Binomial theorems. In this note a theorem on these polynomials is established.

Theorem Let \( n \) be a positive integer,

If \( \cos (4n+1)x = f_{4n+1}(\cos x) \) then \( \sin (4n+1)x = f_{4n+1}(\sin x) \) \( \quad (I) \)

If \( \cos (4n-1)x = g_{4n-1}(\cos x) \) then \( \sin (4n-1)x = -g_{4n-1}(\sin x) \) \( \quad (II) \)

where \( f_{4n+1}(\cdot) \) and \( g_{4n-1}(\cdot) \) are polynomials of degrees \( (4n+1) \) and \( (4n-1) \) respectively.

The results \( (I) \) and \( (II) \) are established in two different ways.

( I ) Proof employing De moivre’s and Binomials:-

When \( n \) is a positive integer, \( \cos(\pi x) \) and \( \sin(\pi x) \) can be expressed as polynomials in \( \sin x \) and \( \cos x \) using Demoivre’s and Binomial theorems.

Consider

\[
\cos(\pi x) + i\sin(\pi x) = e^{i\pi x} = (\cos x + i\sin x)^\pi, \quad \text{where } i = \sqrt{-1}
\]

Expanding the R.H.S of (1) employing Binomial theorem

\[
\cos(4n+1)x + i\sin(4n+1)x = e^{i(4n+1)x} = (\cos x + i\sin x)^{4n+1}
\]

Expanding the R.H.S of (1) employing Binomial theorem

\[
\cos(4n+1)x + i\sin(4n+1)x = \left(\cos x\right)^{4n+1} + i\left(\sin x\right)^{4n+1} + \binom{4n+1}{1}\cos x\sin x + \binom{4n+1}{2}\cos^2 x\sin^2 x + \cdots + i\sum_{k=0}^{\infty} \binom{4n+1}{2k+1}\frac{x^{4n+2k+1}}{(4n+1)!} + \sum_{k=0}^{\infty} \binom{4n+1}{2k}\frac{x^{4n+2k}}{(4n+1)!} + i\sum_{k=0}^{\infty} \binom{4n+1}{2k+1}\frac{x^{4n+2k+1}}{(4n+1)!} + \cdots
\]
Equating real and imaginary parts on both sides of (2), we get

\[
\cos(4n+1)x = \left\{ \begin{array}{l}
\left( \frac{4n+1}{1} \right) (\cos^{4n} x) \sin x - \left( \frac{4n+1}{3} \right) (\cos^{4n-2} x) (\sin^3 x) + \left( \frac{4n+1}{5} \right) (\cos^{4n-4} x) (\sin^5 x) - \ldots \ldots \ldots \\
+ \left( \frac{4n+1}{4n} \right) (\sin^{4n-1} x)
\end{array} \right.
\]

(2)

\[
\sin(4n+1)x = \left\{ \begin{array}{l}
\left( \frac{4n+1}{1} \right) (\cos^{4n} x) \sin x - \left( \frac{4n+1}{3} \right) (\cos^{4n-2} x) (\sin^3 x) + \ldots \ldots \ldots \\
+ \left( \frac{4n+1}{4n} \right) (\cos^{4n-1} x) + (\sin^{4n+1} x)
\end{array} \right.
\]

(3)

Now the expression on the R.H.S of (3) can be written as

\[
\cos (4n+1)x = \left\{ \begin{array}{l}
\left( \frac{4n+1}{1} \right) (\cos^{4n} x) (\sin x) - \left( \frac{4n+1}{3} \right) (\cos^{4n-2} x) (\sin^3 x) + \ldots \ldots \ldots \\
+ \left( \frac{4n+1}{4} \right) (\cos^{4n-1} x) (1-\cos^2 x) - \ldots \ldots \ldots \\
+ \left( \frac{4n+1}{6} \right) (\cos^{4n-3} x) (1-\cos^2 x)^3 \\
\ldots \ldots + (-1)^p \left( \frac{4n+1}{2p} \right) (\cos^{4n-2p+1} x) (1-\cos^2 x)^p + \ldots \ldots \ldots - \left( \frac{4n+1}{4n} \right) (\cos x) (1-\cos^2 x)^{2n}
\end{array} \right.
\]

\[
= \sum_{p=0}^{2n} (-1)^p \left( \frac{4n+1}{2p} \right) (\cos x)^{4n-2p+1} (1-\cos^2 x)^p = f_{4n+1} (\cos x) \ldots (say)
\]

(5)

Also, writing the expression on the R.H.S of expression (4) in the reverse order, we get

\[
\sin (4n+1)x = \left\{ \begin{array}{l}
(\sin^{4n} x) - \left( \frac{4n+1}{4n-1} \right) (1-\sin^2 x) (\sin^{4n-1} x) + \ldots \ldots + (-1)^p \left( \frac{4n+1}{2p+1} \right) (1-\sin^2 x)^{2n-p} (\sin^{2p+1} x) \\
\ldots \ldots + \left( \frac{4n+1}{3} \right) (1-\sin^2 x)^{2n-1} (\sin^3 x) + \left( \frac{4n+1}{1} \right) (1-\sin^2 x)^{2n} (\sin x)
\end{array} \right.
\]

(6)

Noting that

\[
\binom{4n+1}{2k} = \binom{4n+1}{(4n+1)-2k}, \text{ for } k = 0, 1, 2, \ldots, 2n
\]

Then, the expression on the R.H.S of (6) can be rewritten as

\[
\sin (4n+1)x = \left\{ \begin{array}{l}
(\sin^{4n} x) - \left( \frac{4n+1}{2} \right) (1-\sin^2 x) (\sin^{4n-1} x) + \ldots \ldots + (-1)^p \left( \frac{4n+1}{2p} \right) (1-\sin^2 x)^{2n-p} (\sin^{2p+1} x)
\end{array} \right.
\]
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\[
\sum_{k=0}^{2n} (-1)^k \left( \begin{array}{c} 4n+1 \\ 2k \end{array} \right) (\sin x)^{4n-2k+1} (1 - \sin^2 x)^k
\]

(7)

This expression is the same as expression on the R.H.S of (5) with \( \cos x \) replaced by \( \sin x \).

we thus have

\[
\sin (4n+1)x = f_{4n+1}(\sin x)
\]

Hence the result : If \( \cos (4n+1) = f_{4n+1}(\cos x) \) then \( \sin (4n+1)x = f_{4n+1}(\sin x) \) (I)

To establish the result (II)

Again consider

\[
\cos (4n-1)x + i \sin (4n-1)x = (\cos x + i \sin x)^{4n-1}
\]

(8)

Expanding the R.H.S of (8) employing Binomial theorem and equating real and imaginary parts on both sides, we get

\[
\cos (4n-1)x = \sum_{k=0}^{2n} (-1)^k \left( \begin{array}{c} 4n-1 \\ 2k \end{array} \right) (\cos x)^{4n-2k} (1 - \cos^2 x)^{k+1}
\]

(9)

And

\[
\sin (4n-1)x = \sum_{k=0}^{2n} (-1)^k \left( \begin{array}{c} 4n-1 \\ 2k \end{array} \right) (\sin x)^{4n-2k} (1 - \sin^2 x)^{k}
\]

By rewriting the R.H.S of the above in the reverse order and noting that

\[
\left( \begin{array}{c} 4n-1 \\ 2k \end{array} \right) = \left( \begin{array}{c} 4n-1 \\ (4n-1) - 2k \end{array} \right), \quad k = 0,1,2,3,..............2n-1
\]

We note that

\[
\sin (4n-1)x = - (\sin^{4n-1} x) - \sum_{k=0}^{2n} (-1)^k \left( \begin{array}{c} 4n-1 \\ 2k \end{array} \right) (\cos x)^{4n-2k} (\sin x)^{4n-2k+1}
\]

(9)
The expression (10) for $\sin (4n-1)x$ is same as the expression (9) with $\cos x$ with replaced by $\sin x$ but for the change in sign. i.e $\sin (4n+1)x = -g_{4n+1}(\sin x)$.

Hence the result: If $\cos (4n-1)x = g_{4n-1}(\cos x)$ then $\sin (4n-1)x = -g_{4n-1}(\sin x)$ (II)

(II) Proof employing the trigonometric properties of complimentary angels.

i) $\sin \{(4n+1)x\} = \sin \{(4n+1)(\frac{\pi}{2} - y)\}$ where $x = \frac{\pi}{2} - y$

$= \sin \{(2n\pi+1)+(\frac{\pi}{2} - (4n+1)y)\}$

$= \sin \{\frac{\pi}{2}-(4n+1)y\}$

$= \cos (4n+1)y = f_{4n+1}(\cos y)$

$= f_{4n+1}(\cos(\frac{\pi}{2} - y))$

$= f_{4n+1}(\sin x)$

ii) Similarly

$\sin \{(4n-1)x\} = \sin \{(4n-1)(\frac{\pi}{2} - y)\}$

$= \sin \{2n\pi - \frac{\pi}{2} - (4n-1)y\}$

$= \sin \{\frac{\pi}{2} - (4n-1)y\}$

$= -\sin \{\frac{\pi}{2} + (4n-1)y\}$

$= \cos (4n-1)y$

$= -g_{4n-1}(\cos y)$

$= -g_{4n-1}(\cos(\frac{\pi}{2} - x))$

$= -g_{4n-1}(\sin x)$

CONCLUSION

The following results are proved in two ways: Employing De movire’s and Binomial theorems and by using the properties of trigonometric ratios of complimentary angles.

i.e. If $\cos (4n+1)x = f_{4n+1}(\cos x)$ then $\sin (4n+1)x = f_{4n+1}(\sin x)$

$\cos (4n-1)x = g_{4n-1}(\cos x)$ then $\sin (4n-1)x = -g_{4n-1}(\sin x)$

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