Abstract: In this paper, we have defined a new graph called $S(n, m)$-graph for even $m \geq 2m + 2$ and for odd $m > 1$ and found the lower and upper bound for the total chromatic number of $S(n, m)$-graphs. We have also found the total chromatic number of $S(n, 2)$ for all $n \geq 6$ and $S(n, 3)$ for odd $n \geq 7$.

Keywords: Total Coloring, $S(n, m)$-graphs

1. INTRODUCTION

For the past three decades many researchers have worked on total coloring of graphs. Borodin [1] has discussed the total coloring of graphs. Sudha and K.Manikandan [3] have discussed the total coloring and total coloring of prisms $P(n, 1)$. H.P.Yap [4] also has defined and discussed the total coloring of graphs. We have defined a new graph $S(n, m, n \geq 2m + 2, m \geq 3)$ and the definition follows:

The graph $S(n, m)$ consists of $n$ vertices denoted as $v_1, v_2, v_3, ..., v_n$. The edges are defined as follows:

(i) $v_i$ is adjacent to $v_{i+m}$ and $v_n$ is adjacent to $v_1$

(ii) $v_i$ is adjacent to $v_{i+m}$ if $i + m < n$

(iii) $v_i$ is adjacent to $v_{i+n-m}$ if $i + m \geq n$.

This graph is a quartic graph and it is both Eulerian and Hamiltonian. The concept of this type of a new graph was introduced by S.Sudha.

Definition 1: A total coloring is a coloring on the vertices and edges of a graph such that

(i) no two adjacent vertices have the same color

(ii) no two adjacent edges have the same color

(iii) no edge and its end vertices are assigned with the same color.

In this paper, we have considered the graph $S(n, m)$ and obtained the upper and lower bound for the total chromatic number.

2. TOTAL COLORING OF $S(n, m)$-GRAPHS

Theorem 1: The total-chromatic number $\chi_{tc}(S(n, m))$ is 6 for $n \geq 2m + 2$ and odd $m \geq 3$.

Proof: Let $v_1, v_2, ..., v_{n-1}, v_n$ be the vertices of the graph $S(n, m)$ and its edges are defined as

(i) $v_i$ is adjacent to $v_{i+m}$ and $v_n$ is adjacent to $v_1$

(ii) $v_i$ is adjacent to $v_{i+m}$ if $i + m < n$

(iii) $v_i$ is adjacent to $v_{i+n-m}$ if $i + m \geq n$. 

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Let the coloring set of \( S(n, m) \) be the set \( \{1, 2, 3, \ldots \} \).

We define the function \( f_1 \) from \( V(S(n, m)) \) to the set \( \{1, 2, 3, \ldots \} \) as follows:

\[
f_1(v_i) = \begin{cases} 
1, & \text{i - odd, } 1 \leq i \leq n \\
2, & \text{i - even, } 1 \leq i \leq n 
\end{cases}
\]

We define the function \( f_2 \) from \( E(S(n, m)) \) to the set \( \{1, 2, 3, \ldots \} \) as follows:

\[
f_2(v_i, v_{i+1}) = \begin{cases} 
3, & \text{i - odd, } 1 \leq i \leq n - 1 \\
4, & \text{i - even, } 1 \leq i \leq n - 1 
\end{cases}
\]

\[
f_2(v_n, v_1) = 4
\]

\[
f_2(v_i, v_{i+m}) = \begin{cases} 
5, & \text{i - odd, } 1 \leq i \leq n - m \\
6, & \text{i - even, } 1 \leq i \leq n - m 
\end{cases}
\]

\[
f_2(v_i, v_{i+n-m}) = \begin{cases} 
5, & \text{i - even, } 1 \leq i \leq m \\
6, & \text{i - odd, } 1 \leq i \leq m 
\end{cases}
\]

By using the above pattern of coloring, the graph \( S(n, m) \) admits total coloring. The total-chromatic number for \( S(n, m) \), \( \chi_{tc}(S(n, m)) = 6 \).

**Illustration 1:**

![Figure 1. S(12,5)](image)

The graph \( S(12, 5) \) consists of 12 vertices \( v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12} \) which are assigned with the colors \( 1, 2, 1, 2, 1, 2, 1, 2, 1, 2 \) respectively. The outer edges \( v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}, v_{10}v_{11}, v_{11}v_{12} \) and \( v_{12}v_{11}, v_{11}v_4, v_4v_9, v_9v_{10}, v_{10}v_{11}, v_{11}v_{12}, v_{12}v_5, v_5v_{10}, v_{10}v_3 \), are assigned with colors \( 3, 4, 3, 4, 3, 4, 3, 4, 3, 4, 3, 4 \) and the inner edges \( v_1v_6, v_6v_{11}, v_3v_8 \) are assigned with colors \( 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6 \) respectively. The total-chromatic number of \( S(12, 5), \chi_{tc}(S(12, 5)) = 6 \).

**Theorem 2:** The total-chromatic number \( \chi_{tc}(S(n, 2)) \) \((n \geq 6)\) is 5 for \( n \equiv 0 \pmod{6} \) and is 6 for \( n \equiv 0 \pmod{6} \).

**Proof:** Let \( v_1, v_2, \ldots, v_{n-1}, v_n \) be the vertices of the graph \( S(n, 2) \) and its edges be denoted by \( \{v_iv_{i+1}, v_{i+2}, \ldots, v_{i+n-2}, v_nv_1\} \) for \( i = 0, 1, \ldots, n-2 \).

Let \( f_1 \) be a function that maps \( V(S(n, 2)) \) to the set \( \{1, 2, 3, \ldots\} \) and \( f_2 \) be a function that maps \( E(S(n, 2)) \) to the set \( \{1, 2, 3, \ldots\} \) in such a way that \( f_1 \) and \( f_2 \) satisfy the condition of total coloring.

There are six cases:

(i) \( n \equiv 0 \pmod{6} \)
Total Coloring of $S(n, m)$-Graph

(ii) $n \equiv 2 \pmod{6}$
(iii) $n \equiv 4 \pmod{6}$
(iv) $n \equiv 1 \pmod{6}$
(v) $n \equiv 3 \pmod{6}$
(vi) $n \equiv 5 \pmod{6}$

**Case (i):** Let $n \equiv 0 \pmod{6}$

\[
f_1(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n 
\end{cases}
\]

\[
f_2(v_iv_{i+1}) = \begin{cases} 
4, & i - \text{odd}, 1 \leq i \leq n - 1 \\
5, & i - \text{even}, 1 \leq i \leq n - 1 
\end{cases}
\]

$f_2(v_{n-1}v_n) = 2$

$f_2(v_{i}v_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n - 2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $S(n, 2)$ for $n \equiv 0 \pmod{6}$ admit total coloring.

The total-chromatic number of \(S(n, 2), \chi_{tc}(S(n, 2)) = 5\).

**Case (ii):** Let $n \equiv 2 \pmod{6}$

\[
f_1(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n - 2 \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n - 2 \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n - 2 
\end{cases}
\]

$f_1(v_{n-1}) = 4$

$f_1(v_n) = 5$

\[
f_2(v_iv_{i+2}) = \begin{cases} 
6, & i - \text{odd}, 1 \leq i \leq n - 3 \\
5, & i - \text{even}, 1 \leq i \leq n - 3 
\end{cases}
\]

$f_2(v_{n-2}v_{n-1}) = 1$

$f_2(v_{n-1}v_n) = 2$

$f_2(v_nv_1) = 3$

$f_2(v_iv_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n - 2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $S(n, 2)$ admit total coloring.

The total-chromatic number of \(S(n, 2), \chi_{tc}(S(n, 2)) = 6\).

**Case (iii):** Let $n \equiv 4 \pmod{6}$

\[
f_1(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n - 1 \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n - 1 \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n - 1 
\end{cases}
\]

$f_1(v_n) = 4$

\[
f_2(v_iv_{i+2}) = \begin{cases} 
5, & i - \text{odd}, 1 \leq i \leq n - 1 \\
6, & i - \text{even}, 1 \leq i \leq n - 1 
\end{cases}
\]

$f_2(v_nv_1) = 6$

$f_2(v_iv_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n - 2$

$f_2(v_{n-1}v_1) = f_1(v_n)$
By using the above pattern, the graph $S(n, 2)$ admit total coloring.

The total-chromatic number of $S(n, 2)$, $\chi_{tc}(S(n, 2)) = 6$.

Case (iv): Let $n \equiv 1(\text{mod } 6)$

$$f_1(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1(\text{mod } 3), 1 \leq i \leq n - 1 \\
2, & \text{for all } i \equiv 2(\text{mod } 3), 1 \leq i \leq n - 1 \\
3, & \text{for all } i \equiv 0(\text{mod } 3), 1 \leq i \leq n - 1 
\end{cases}$$

$$f_1(v_n) = 4$$

$$f_2(v_i v_{i+1}) = \begin{cases} 
5, & i - \text{odd}, 3 \leq i \leq n - 2 \\
6, & i - \text{even}, 4 \leq i \leq n - 1 
\end{cases}$$

$$f_2(v_1 v_2) = 6$$

$$f_2(v_2 v_3) = 4$$

$$f_2(v_n v_1) = 5$$

$$f_2(v_i v_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n - 2$$

$$f_2(v_{n-1} v_1) = f_1(v_n)$$

By using the above pattern, the graph $S(n, 2)$ admit total coloring.

The total-chromatic number of $S(n, 2)$, $\chi_{tc}(S(n, 2)) = 6$.

Case (v): Let $n \equiv 3(\text{mod } 6)$

$$f_1(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1(\text{mod } 3), 1 \leq i \leq n - 1 \\
2, & \text{for all } i \equiv 2(\text{mod } 3), 1 \leq i \leq n - 1 \\
3, & \text{for all } i \equiv 0(\text{mod } 3), 1 \leq i \leq n - 1 
\end{cases}$$

$$f_1(v_n) = 6$$

$$f_2(v_i v_{i+1}) = \begin{cases} 
4, & i - \text{odd}, 1 \leq i \leq n - 1 \\
5, & i - \text{even}, 1 \leq i \leq n - 1 
\end{cases}$$

$$f_2(v_n v_1) = 6$$

$$f_2(v_i v_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n - 2$$

$$f_2(v_{n-1} v_1) = f_1(v_n)$$

By using the above pattern, the graph $S(n, 2)$ admit total coloring.

The total-chromatic number of $S(n, 2)$, $\chi_{tc}(S(n, 2)) = 6$.

Case (vi): Let $n \equiv 5(\text{mod } 6)$

$$f_1(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1(\text{mod } 3), 1 \leq i \leq n - 2 \\
2, & \text{for all } i \equiv 2(\text{mod } 3), 1 \leq i \leq n - 2 \\
3, & \text{for all } i \equiv 0(\text{mod } 3), 1 \leq i \leq n - 2 
\end{cases}$$

$$f_1(v_{n-1}) = 4$$

$$f_1(v_n) = 5$$

$$f_2(v_i v_{i+1}) = \begin{cases} 
6, & i - \text{odd}, 1 \leq i \leq n - 2 \\
5, & i - \text{even}, 1 \leq i \leq n - 2 
\end{cases}$$

$$f_2(v_{n-1} v_n) = 2$$

$$f_2(v_n v_1) = 3$$

$$f_2(v_i v_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n - 2$$

$$f_2(v_{n-1} v_1) = f_1(v_n)$$
Total Coloring of $S(n,m)$-Graph

By using the above pattern, the graph $S(n,2)$ admit total coloring.

The total-chromatic number of $S(n,2)$, $\chi_{tc}(S(n,2)) = 6$.

**Illustration 2:**

![Figure 2. S(8,2)](image)

The graph $S(8,2)$ consists of 8 vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ which are assigned with the colors 1,2,3,1,2,3,4,5 respectively. The outer edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8$ and $v_8v_1$ are assigned with colors 6,5,6,5,6,1,2,3 and the inner edges $v_1v_3, v_3v_5, v_5v_7, v_7v_1, v_2v_4, v_4v_6, v_6v_8, v_8v_2$ are assigned with colors 2,1,3,5,3,2,4,1 respectively.

The total-chromatic number of $S(8,2)$, $\chi_{tc}(S(8,2)) = 6$.

**Theorem 3:** The total-chromatic number $(\chi_{tc}(S(n,3)))$ is 7 for $n \equiv 0$ (mod 6) and is 6 for $n \equiv 0$ (mod 6).

**Proof:** Let $v_1, v_2, \ldots, v_{n-1}, v_n$ be the vertices of the graph $S(n,3)$ and its edges be denoted by $(v_1v_{i+1}), (v_{i}v_{i+2}), (v_1v_{i+n-2})$ for $i = 1,2,3,\ldots$ and $(v_nv_1)$.

Let $f_1$ be a function that maps $V(S(n,3))$ to the set $\{1,2,3,\ldots\}$ and $f_2$ be a function that maps $E(S(n,3))$ to the set $\{1,2,3,\ldots\}$ in such a way that $f_1$ and $f_2$ satisfy the condition of total coloring.

**Case(i):** For odd $n \geq 7$ and $n \not\equiv 0$ (mod 3)

$f_1(v_i) = \begin{cases} 1, & \text{for all } 1 \leq i \leq n-4, i - \text{odd} \\ 2, & \text{for all } 1 \leq i \leq n-3, i - \text{even} \\ 3, & \text{for all } i = n - 2, n \end{cases}$

$f_1(v_{n-1}) = 4$

$f_2(v_1v_{i+1}) = \begin{cases} 3, & \text{for all } 1 \leq i \leq n-4, i - \text{odd} \\ 4, & \text{for all } 1 \leq i \leq n-3, i - \text{even} \end{cases}$

$f_2(v_{n-2}v_{n-1}) = 2$

$f_2(v_{n-1}v_n) = 1$

$f_2(v_nv_1) = 2$

The edges of the form $f_2(v_1v_{i+3})$ for $i = 1,2,3,\ldots, (n-3)$. Takes the coloring pattern as 5,6,5,6,\ldots,5,6.

The last three of the edges are colored given below.

$f_2(v_{n-8}v_{n-3}) = 5$

$f_2(v_{n-5}v_{n-2}) = 1$

$f_2(v_{n-2}v_1) = 6$

By using the above pattern, the graph $S(n,3)$ admit total coloring.
The total-chromatic number of $S(n, 3)$, $\chi tc(S(n, 3)) = 6$.

**Case (ii): For odd $n \geq 9$ and $n \equiv 0(mod 3)$**

- $f_1(v_i) = \begin{cases} 
4, \text{for all } 1 \leq i \leq n - 4, i - odd \\
5, \text{for all } 1 \leq i \leq n - 3, i - even
\end{cases}$
- $f_1(v_{n-2}) = 3$
- $f_1(v_{n-1}) = 4$
- $f_1(v_n) = 3$

- $f_2(v_{i}v_{i+1}) = \begin{cases} 
3, \text{for all } 1 \leq i \leq n - 4, i - odd, 1 \leq i \leq n - 3 \\
5, \text{for all } 1 \leq i \leq n - 3, i - even, 1 \leq i \leq n - 3
\end{cases}$
- $f_2(v_{n-2}v_{n-1}) = 2$
- $f_2(v_{n-1}v_n) = 1$
- $f_2(v_nv_1) = 2$

- $f_2(v_{i}v_{i+3}) = \begin{cases} 
5, \text{ for odd, } 1 \leq i \leq n - 3 \\
6, \text{ for even, } 1 \leq i \leq n - 3
\end{cases}$

- $f_2(v_{i}v_{i+n-3}) = 7, 1 \leq i \leq 3$.

Now with this type of coloring, the graph $S(n, 3)$ is total coloring.

The total-chromatic number of $S(n, 3)$, $\chi tc(S(n, 3)) = 6$.

**Illustration 3:**

![Figure 3](image)

**Figure 3. $S(11,3)$**

The graph $S(11,3)$ consists of 11 vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}$ which are assigned with the colors $1, 2, 1, 2, 1, 2, 3, 4, 3$ respectively. The outer edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}, v_{10}v_{11}$ and $v_{11}v_1$ are assigned with colors $3, 4, 3, 4, 3, 4, 3, 2, 1, 2$ and the inner edges $v_1v_4, v_4v_7, v_7v_{10}, v_{10}v_2, v_2v_5, v_5v_8, v_8v_{11}, v_{11}v_3, v_3v_6, v_6v_9, v_9v_1$ are assigned with colors $5, 6, 5, 6, 5, 6, 5, 6, 5, 6$. The total-chromatic number of $S(11,3)$, $\chi tc(S(11,3)) = 6$.

### 3. Conclusion

We have found that the lower and upper bound for the total chromatic number of $S(n, m)$, in general, satisfies $5 \leq \chi tc(S(n, m)) \leq 7$. The total-chromatic number for $S(n, m)$ when $m$ takes the value 2 and 3 are also discussed.

### References


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