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Optimum Linear Stochastic Control with Input-Output Equations

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Abstract: *The paper presents methods for solving the discrete time lumped and time invariant parameter optimum linear stochastic control problem on finite and infinite intervals with I/O equations. The finite horizon optimum control problem may be solved through a series of two-stage optimizations, the infinite horizon optimum control problem through optimization and limit value calculation. An example shows the applicability of the method.*

Keywords: *optimum control, stochastic control, tracking control, finite horizon control, infinite horizon control, steady state feedback gain.*

1. Introduction

Design of control may be based on fitting the outputs to the references, which sometimes are known in advance, another time they may be estimated. The optimization in the paper is carried out through the optimization method "optimized stochastic trajectory / output sequence tracking" (OSTT), which is extension of the deterministic optimization method "optimized trajectory tracking" (OTT) [4], [5]. The finite horizon control problem may be solved through a sequence of optimization on a section of two control stages. Application of two-stage optimization is based on the fact that the optimum on a finite horizon in general isn't equal with the sum of one-stage optimums, and OTT and OSTT show how to compute it. Solution of the optimum infinite horizon control problem (computation of the steady state optimum feedback gain) can be accomplished through optimizationon on a section of two control stages and limit value calculation. Stochastic control can be computed similarly as deterministic one, taking into consideraton the differences.

2. Model, Performance Index and Prediction

2.1. Model

The plant to be controlled is supposed to be given with a stochastic model, e.g. with the ARMAX transfer function model (ARMA autoregressive moving average model [3] with exogenous inputs), which for one step delay in the control signal is

$$G(q^{-1})\,y(t) = q^{-1}\,H(q^{-1})\,u(t) + L(q^{-1})\,e(t). \qquad (1)$$

In (1) $G(q^{-1})$, $H(q^{-1})$ and $L(q^{-1})$ are polynomials of appropriate dimensions and q^{-1} is the backward shift operator [1],

$$G(q^{-1})=1+g1q^{-1}+\ldots+gngq^{-ng}, \qquad (2)$$

$$H(q^{-1})=h_1+\ldots+h_{nh}q^{-nh}, \qquad (3)$$

$$L(q^{-1})=1+l_1q^{-1}+\ldots+l_{nl}q^{-nl}, \qquad (4)$$

where t is time, $y(t)$ is the output, u(t) is the input, $e(t)$ is white noise sentence of independent random variables with zero mean.

2.2. Performance index

For finite horizon linear I/O optimum stochastic tracking and references known in advance, the performance index (PI) may be

$$F_{1,N} = \sum_{j=1}^{N} \{[\hat{y}(t+j) - r(t+j)]^2 + \lambda u^2(t+j-1)\}. \quad (5)$$

In (5) r is the reference, N is the length of horizon in steps. For two-stage optimizations the corresponding PI may be

$$F_{i+1,i+2} = \sum_{j=i+1}^{i+2} \{[\hat{y}(t+j) - r(t+j)]^2 + \lambda u^2(t+j-1)\}. \quad (6)$$

In (5), (6) λ is control weight. Additional constraints may be treated e.g. through the mthod of Lagrange multipliers.

2.3. Prediction

Assume that the last measurement happend at t. The one-stage ahead prediction from (1) is

$$\hat{y}(t+1/t) = \frac{q^{-1}H(q^{-1})}{G(q^{-1})}u(t+1) + \frac{L(q^{-1})}{G(q^{-1})}e(t+1). \quad (7)$$

$e(t)$, $e(t$-1)...can be computed as the difference between the measured and estimated outputs. If to the estimation $e(t+1)=0$ is applied, the estimation is suboptimal. However, when feedback control is used, the control signal is computed at the beginning of a two-stage section [6], and $e(t+1)=0$ has to be assumed. (7) is a simple algorithm for computations through not optimum predictions. However, in practice not necessarily appropriate to minimize the variance of output [1]. The one-stage ahead prediction is convenient for computations which are based on a series of two-stage optimizations. Feedback solution can be obtained if $u(i+1)$ and $y(i+1)$ are eliminated through system equations and expected value of the unknown $u(i)$ is taken.

Another possibility is d-step ahead optimum minimum variance output prediction [1], [2],

$$\hat{y}(t+d/t) = \{DH/L\}u(t) + \{F/L\}y(t), \quad (8)$$

where D, H, L, F are polynomials of appropriate dimensions, , d=deg G - deg H and F mut satisfy the polynomial relationship [1], [2]

$G(q^{-1})D(q^{-1})+(q^{-d})F(q^{-1})=L(q^{-1})$.

The control signal can be obtained from solution of the control problem with optimization.

3. FINITEAND INFINITE HORIZON OPTIMUM CONTROL

3.1. Finite Horizon Optimum Stochastic Tracking

The solution for the optimum control signals can be obtained on the whole horizon with (6) through a series of optimizations. It is asumed that estimates have been obtained on all the past outputs. When applying the two-stage optimizations, optimum value of the control signal on the second stage is obtained in function of the control signal on the first stage. However, a question is the $u(t0)$opt optimum value of the control signal at the first stage of horizon. Solution of the corresponding model predictive control (MPC) problem gives the optimum output sequence between the initial point and end point [5]. When solving the MPC problem, it is assumed that all the future disturbances are zero [5]. Since future disturbances can't be taken into consideration in advance, $u(t0)$opt is common both for MPC and optimum stochastic tracking. Optimum value of $\hat{y}(t_0+1)$ can be estimated with this $u(t_0)_{opt.}$ and the optimum control signals can be computed on the whole horizon.

3.1.1. Computation of the optimum initial control signal

The computation can be achieved with (5) and (6), with $e(t+i)=0$, i=1,2,..Suppose an optimum output sequence in function of a selected $u(t_0)$ starting control signal has been computed on the time interval $[t+1, t+i+1]$. The next unknown optimum output can be computed through the optimization

$$\min F_{i+1,i+2} = \min_{u(i+1)} = \sum_{j=i+1}^{i+2} f\{y(t+j), r(t+j), u(t+j-1)\}. \tag{9}$$

The necessary condition for minimum is

$$\frac{\partial F_{i+1,i+2}}{\partial u(t+i)} = 0. \tag{10}$$

For evaluation of (10), $y(t+i+1)$ and $y(t+i+2)$ are substituted with system equations. Although $u(t+i)$ is known, here it is considered as unknown, and after derivation its value is substituted. The unknown control signal can be computed as

$$u(t+i+1)=f_i\{u(t+i),u(t+i-1),...y(t+i),y(t+i-1),...r(t+i+1),\ r(t+i+2)\}. \tag{11}$$

With (9) and the formerly computed values $\hat{y}(t+i+2)$ can be estimated, and the whole optimum output sequence can be computed similrly with OSTT. At the the start of computations an estimation has to be made on the first stage, e.g.

$$\hat{y}(1) \approx r(1). \tag{12}$$

With completing the two-stage optimization N-1 times on an N-stage horizon,$\hat{y}(N)\neq r(N)$ is reached. The final solution can be obtained through iterations. For application of the Newton method. deviation of the final value for a SISO system can be written as

$$r(t+N) - \hat{y}(t+N) \approx \frac{\partial \hat{y}(t+N)}{\partial u(t)} \Delta u(t). \tag{13}$$

From (13)

$$\Delta u(t) \approx \frac{r(t+N) - \hat{y}(t+N)}{\partial \hat{y}(t+N) / \partial u(t)}. \tag{14}$$

The new starting value for computation is

$$u'(t) \approx u(t) + \Delta u(t). \tag{15}$$

Derivation in (12)can be approximated as

$$\frac{\partial \hat{y}(t+N)}{\partial u(t)} \approx \frac{\{\hat{y}((t+N), u(t+h)) - \{\hat{y}((t+N), u(t))\}}{h}, \tag{16}$$

where h i an appropriate small value. The Newton method can be generalized for multivariable systems through use of the Jacobian matrix. If the receding horizon principle is used, only variables on the first stage are kept, the others are discarded, and the procedure is repeated at the next sampling instant. The design method can be generalized for multivariable systems through applying vectors instead of scalars.

3.2. Infinite Horizon Stochastic Tracking

The control method descripted in section 3.1.1. can be used on infinite horizon, too. In this case the starting $u(t_0)$ control signal is selected, e.g. from (12), and $u(t_0)_{opt}$ is not estimated. However, new two-stage optimization has to be made at each section. The computation can be simplified, if the steady state feedback gain is used to the control. (11) can be separated for a dynamic feedback and a feedforward prefilter depending on r(t+i+1) and r(t+i+2). From (11) the control signal on the first stage of a two-stage section can be expressed. Assume that $r(t+i+1)=r(t+i+2)=0$.

Then

$$u(t+i+1) = f_2\{u(t+i), u(t+i-1), \ldots y(t+i), y(t+i-1), \ldots\}. \tag{17}$$

By eliminating $u(t+i+1)$ and $y(t+i+1)$ with system equations,

$$u(t+i) = f_3\{u(t+i-1), u(t+i-2), \ldots y(t+i+2), y(t+i), y(t+i-1), \ldots\}. \tag{18}$$

(18) gives a noncausal transfer function with which the loop isn't stable in general. The DC feedback gain can be computed from (18) with the assumptions

$$y(t+i+2) = y(t+i) = y(t+i-1) = \ldots, \tag{19}$$

$$u(t+i) = u(t+i-1) = u(t+i-2) =, \ldots \tag{20}$$

However, there is less guarantee for stability or good quality of control, if the feedback is made in the above way,

4. Example

Consider a plant given with the model

$$y(i+1)=ay(i)+bu(i)+ce(i), \tag{21}$$

where $y(i)$ – output, a, b, c – parameters, $e(i)$ – sequence of independent random variables with zero expectation and $\sigma^2 = 0.075$ variance. To the simulation $a = 0.9$, $b = 0.8$ and $c = 0.95$ has been selected. The PI for two stages is given as

$$F_{i+1}^{i+2} = \sum_{j=i+1}^{i+2} \{(r(j) - \hat{y}(j))^2 + \lambda u^2(j-1)\}. \tag{22}$$

The necessary condition for minimum is given in (10). With (10), (21), (22)

$$u(i+1) = -\{b^2(1+a^2)+\lambda\}u(i)/(ab^2) - (1+a^2)y(i)/b + r(i+1)/(ab) + r(i+2)/b - c(1+a^2)e(i)/(ab). \tag{23}$$

$e(i)$ can be computed as the difference between the measurement and estimation for $y(i)$. (23) Is the feed forward solution. The feedback solution can be obtained if $u(i+1)$ and $\hat{y}(i+1)$ are expressed by help of system equations and substituted into (23) as

$$u(i) = \frac{b}{b^2+\lambda}\{r(i+1) + ar(i+2) - ay(i) - a\hat{y}(i+2) - ce(i)\}. \tag{24}$$

If only feedforward solution is sought, (21) may include for better result an additional term $e(i+1)$,which is available, when computing $u(i+1)$. However, to the feedback solution the control signal has to be given out one step earlier, consequently $e(i+1)$ is not available, when computing $u(i)$. Therefore, (23) is the feedforward solution to compute the feedback. Simulation results with λ=0.1 can be seen on Fig. 1:

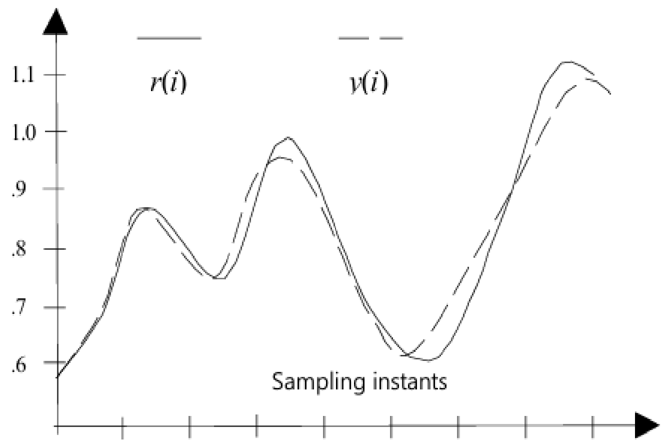

FIGURE1. *Simulation result with receding horizon control*

This problem has been solved with the optimum output estimation (8), too, and the computation can be seen in [6]. The obtained feedback solution is

$$u(i) = \frac{b}{b^2+\lambda}\{r(i+1) + ar(i+2) - ay(i) - a\hat{y}(i+2)\}. \tag{25}$$

(25) is at the same time solution of an MPC problem with the above conditions.

5. Conclusions

The paper shows that control design based on two-stage optimizations, both on finite and infinite horizons, can be extended for optimum linear stochastic tracking with I/O equations, too, and describes the proposed design method. Results of simulations are also satisfactory, agreeing with former findings, that the two-stage optimum tracking is more accurate that the one-stage one.

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