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Lagrangian Equations on Three-Dimensional Para-Kenmotsu manifolds

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Abstract: *In this study we concluded the Lagrangian equations on (M^3,ϕ.ξ,η,g), The Three-Dimensional almost Para-Kenmotsu manifolds, have been derived also important applications of Euler-Lagrange mechanical systems . Finally achieved that Three-Dimensional almost Para - Kenmotsu manifolds have this systems in Mechanics and Physical Fields as well as in differential geometry.*

Keywords: *Para-Kenmotsu manifolds, three – dimensional Almost Kenmotsu manifolds, Lagrangian Equations.*

1. INTRODUCTION

The geometric study of dynamical systems is an important chapter of contemporary mathematics due to its applications in Mechanics, Theoretical Physics.

There are also a large number of studies on this subject, for example **K. Srivastava** submitted On Para-Kenmotsu manifolds[1], and **De**, **Bejan, and DE.Mondal** obtained On 3-dimensional normal Para-contact geometry [2] [3].

Its is some important work for examples [4],[5][6],[7][8],[9] In this paper we will study Kenmotsu manifolds

In this paper, we Euler-Lagrange Equations on Three-Dimensional almost Kenmotsu manifolds. After Introduction in Section 1, we consider Historical Background paper basic. Section 2 deals with the study preliminary . Section 3 is devoted to study 3. three – dimensional Almost Kenmotsu manifolds .Section 4 is devoted to study . Euler-Lagrange Equations on Three-Dimensional almost Kenmotsu manifolds

2. PRELIMINARY

In this In this preliminary chapter, we recall basic definitions, results and formulas which we shall use in the subsequent chapters of the paper

Kenmotsu manifolds 2.1

Let M^{2n+1} be a $(2n+1)$-dimensional smooth differentiable manifold $(\phi.\xi,\eta,g)$ be an almost contact Riemannian manifold .where $\emptyset$ is a(1-1) tenser field η is a 1- form and the Riemannian metric .It well known that

$$\phi(\xi)=0$$

$$\eta(\phi(x))=0 \qquad and \qquad \eta(\xi)=1$$

$$\phi^2(X)=-X+\eta(X)\xi \quad , \quad \phi^2=-1+\eta\otimes\xi$$

$$g(X,\xi)=\eta(X)$$

$$g(\phi x,\phi y)=g(x,y)-\eta(x)\eta(y)$$

$$rank\ \phi=n-1$$

The fundamental 2- form of an almost contact metric manifolds is defined by

$$\phi(x, y) = g(x, \phi y)$$

Lemma 2.2 [2] If ω, θ and k −form be respectively then$d\omega \wedge d\theta = -d\theta \wedge d\omega$

(i) $d^2 = d \circ d = 0$

(ii) $d(\omega\wedge\psi) = d\omega\wedge\psi - d\psi\wedge\omega$

Lemma 2.3 [2] Suppose $(U, x_1, \ldots, x_n)$ is a chart on a manifold. Then $\left(\frac{\partial x^j}{\partial x^i}\right) = \delta^i_j = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$

3. Para-Kenmotsu Manifold

Let M_n be an -dimensional differentiable manifold equipped with structure tensors(Φ, ξ, η) , where Φ is a tensor of type (1, 1) ξ , is a vector field η, is a 1-form such that

$$\eta(\xi) = 1$$

$$\phi^2(X) = X - \eta(X)\xi \quad , \quad \bar{X} = \phi X$$

Then M_n is called an almost para contact manifold. Let g be the Riemannian metric satisfying such that, for all vector fields X and Y on M_n

$$g(X, \xi) = \eta(X)$$

$$g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y)$$

$$\eta(\phi(x)) = 0 \qquad and \quad rnak\, \phi = n - 1$$

Then the manifold [10] is said to admit an almost Para contact Riemannian structure (Φ, ξ, η, g) . A manifold $M_n$of dimension n with Riemannian metric g admitting a tensor field Φ of type (1, 1), a vector field ξ and a 1-form η satisfying (2.1), (2.3) along with

$$(\nabla_X\eta)Y - (\nabla_Y\eta)X = 0$$

$$(\nabla_X\nabla_Y\eta)Z = [-g(X, Y) + \eta(X)\eta(Z)\eta(Y) + [-g g(x, y) - \eta(x)\eta(y)]\eta(Z)$$

$$\nabla_X\xi = \phi^2(X) = X - \eta(X)\xi$$

$$(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

is called a Para-Kenmotsu manifold or briefly p-Kenmotsu manifold [13], where ∇ is the covariant differentiation with respect to the metric . g

Let (M_n, g)be an -dimensional Riemannian manifold admitting a tensor field of type (1, 1), a vector field ϕ and a 1-form η satisfying

$$(\nabla_X\eta)Y = g(X, Y) - \eta(Y)\phi X$$

$$g(X, Y) = \eta(X) \text{ and } (\nabla_X\eta)Y = \phi(\bar{X}, Y)$$

where ϕ is an associate of. ϕ

Theorem 3.1. If the metric of a three-dimensional f-para Kenmotsu manifold is a Ricci solution, then the manifold is of constant negative curvature f^2.

Theorem 3.2. If in a three-dimensional f -Para Kenmotsu manifold, the metric is Ricci solution and ξ is point wise collinear with ξ, then V is constant multiple of V and consequently ξ is complete.

4. Three – Dimensional Para - Almost Kenmotsu Manifolds

Let three-dimensional manifold $M = f(x, y, z) \in R^3, z \neq 0$; where (x, y, z) are the standard coordinates in R^3: The vector fields

$$e_1 = \frac{\partial}{\partial x} \quad , \quad e_2 = \frac{\partial}{\partial y} \quad , \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M: Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \aleph(M)$.

Proposition 3.4

Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = e_2$$
$$\phi(e_2) = e_1$$
$$\phi(e_3) = 0$$

Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1\,;\ \phi^2(z) = -Z + \eta(Z)e_3\ ;$$
$$g(\phi Z; \phi W) = g(Z; W) - \eta(Z)\eta(W);$$

for any $Z, W \in \phi(M)$: Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M: Now, by direct computations we obtain

$$[e_1, e_3] = 0 \quad ; [e_2, e_3] = e_2 \ , \qquad [e_1, e_3] = e_1$$

Proposition 4.1

The vector fields

$$e_1 = \frac{\partial}{\partial x} \quad , \qquad e_2 = \frac{\partial}{\partial y} \qquad , \qquad e_3 = \frac{\partial}{\partial z}$$

If ϕ is defined a Para - complex manifold$\mathcal{M}$then $\phi^2 = \phi \circ \phi = 1$

Proof.

$$\phi^2(e_1) = \phi(\phi(e_1)) = \phi(\ e_2) = \phi(e_2) = e_1 = 1$$
$$\phi^2(e_2) = \phi(\phi(e_2)) = \phi(\ e_1) = \phi(e_1) = e_2 = 1$$
$$\phi^2(e_3) = \phi(\phi(e_3)) = \phi(\ 0) = \phi(0) = 0$$

As can ϕ^2 is 1 (complex) or 0

5. Lagrangian Equations on Three-Dimensional almost Para- Kenmotsu manifolds

In this section, we shall obtain the version Euler-Lagrange equations for classical mechanics structured with Three-Dimensional Para- Kenmotsu manifolds introduced in

Let semispray be a vector field as follows

$$\xi = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \quad , \quad x = \dot{x}\ , \quad y = \dot{y}\ , \quad z = \dot{z}$$

By Liouville vector field on Three-Dimensional almost Para- Kenmotsu manifolds space form $(M^3, \phi. \xi, \eta, g)$, we call the vector field determined by $V = \phi\xi$ and calculated by

$$\phi\xi = \phi\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) = X\phi\left(\frac{\partial}{\partial x}\right) + Y\phi\left(\frac{\partial}{\partial x}\right) + Z\phi\left(\frac{\partial}{\partial x}\right)$$

$$\phi\xi = X\frac{\partial}{\partial y} + Y\frac{\partial}{\partial x} + Z(0) = X\frac{\partial}{\partial y} + Y\frac{\partial}{\partial x} + Z(0)$$

$$\phi\xi = X\frac{\partial}{\partial y} + Y\frac{\partial}{\partial x}$$

Denote T by the kinetic energy and P by the potential energy of mechanics system on Three-Dimensional almost Para- Kenmotsu manifolds. Then we write by $L = T - P$ Lagrangian function and by

$$E_L = V(L) - L$$

the energy function associated L. Operator i_J defined by

$$i_\phi : \Lambda^2 M^3 \longrightarrow \Lambda^1 M^3$$

is called the interior product with ϕ, or sometimes the insertion operator, or contraction by ϕ.The exterior vertical derivation d_ϕ is defined by

$$d_\phi = [i_\phi, d] = i_\phi d - d i_\phi$$

where d is the usual exterior derivation. For almost product structure ϕ determined by the closed Three-Dimensional almost Para-Kenmotsu manifolds form is the closed 2-form given by

$$\phi_L = -d d_\phi L$$

such that

$$d_\phi = X\frac{\partial}{\partial y} + Y\frac{\partial}{\partial x}$$

$$d_\phi L = \left(X\frac{\partial}{\partial y} + Y\frac{\partial}{\partial x}\right)L = X\frac{\partial L}{\partial y} + Y\frac{\partial L}{\partial x}$$

Thus we get

$$\phi_L = -d(d_\phi L) = -d\left(X\frac{\partial L}{\partial y} + Y\frac{\partial L}{\partial x}\right)$$

$$\phi_L = -X\frac{\partial^2 L}{\partial x\,\partial y}dx\wedge dx + X\frac{\partial^2 L}{\partial y\,\partial y}dy\wedge dy + Y\frac{\partial^2 L}{\partial x\,\partial x}dx\wedge dy + Y\frac{\partial^2 L}{\partial y\,\partial y}dy\wedge dy + Z\frac{\partial^2 L}{\partial x\,\partial z}dx\wedge dz - -Z\frac{\partial^2 L}{\partial y\,\partial z}dy\wedge dz$$

Because of the closed Three-Dimensional almost Kenmotsu manifolds form ϕ_L on Three-Dimensional almost Para-Kenmotsu manifolds space form $(M^3, \phi.\xi, \eta, g)$ is para-symplectic structure, one may obtain

$$E_L = X\frac{\partial L}{\partial Y} + Y\frac{\partial L}{\partial x} - L$$

Considering (0,1) we calculate

$$dE_L = d\left(X\frac{\partial L}{\partial Y} + Y\frac{\partial L}{\partial x} - L\right)$$

$$dE_L = X\frac{\partial^2 L}{\partial x\,\partial y}dx + y\frac{\partial^2 L}{\partial y\,\partial y}dx + X\frac{\partial^2 L}{\partial x\,\partial x}dy + Y\frac{\partial^2 L}{\partial y\,\partial x}dy - \frac{\partial L}{\partial x}dx - Z\frac{\partial^2 L}{\partial x\,\partial z}dx - Z\frac{\partial^2 L}{\partial y\,\partial z}dy - \frac{\partial L}{\partial y}dy$$

Taking care of $i_\xi \phi_L = dE_L$, we have

$$X\frac{\partial^2 L}{\partial x\,\partial y}dx + y\frac{\partial^2 L}{\partial y\,\partial y}dx + Z\frac{\partial^2 L}{\partial y\,\partial z}dx - \frac{\partial L}{\partial x}dx + X\frac{\partial^2 L}{\partial x\,\partial x}dy + Y\frac{\partial^2 L}{\partial y\,\partial x}dy + Z\frac{\partial^2 L}{\partial x\,\partial z}dy + \frac{\partial L}{\partial y}dy = 0$$

$$\frac{\partial L}{\partial y}\left(X\frac{\partial}{\partial x}dx + y\frac{\partial}{\partial y}dx + Z\frac{\partial}{\partial z}dx\right) + \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial x}\left(X\frac{\partial}{\partial x}dy + Y\frac{\partial}{\partial y}dy + Z\frac{\partial}{\partial z}dy\right) - \frac{\partial L}{\partial y}dy = 0$$

$$\frac{\partial L}{\partial y}\left(X\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)dx + \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial x}\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)dy - \frac{\partial L}{\partial y}dy = 0$$

$$\left[\frac{\partial L}{\partial y}\left(X\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) + \frac{\partial L}{\partial x}\right]dx + \left[\frac{\partial L}{\partial x}\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) - \frac{\partial L}{\partial y}\right]dy = 0$$

If the curve $\alpha : I \subset R \rightarrow M^3$ be integral curve of ξ,

$$\alpha = \frac{\partial}{\partial t} = X\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}$$

which satisfies

$$\left[\frac{\partial}{\partial t}\frac{\partial L}{\partial y} + \frac{\partial L}{\partial x}\right]dx + \left[\frac{\partial}{\partial t}\frac{\partial L}{\partial x} - \frac{\partial L}{\partial y}\right]dy = 0$$

it follows equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial y} + \frac{\partial L}{\partial x} = 0 \ , \quad \frac{\partial}{\partial t}\frac{\partial L}{\partial x} - \frac{\partial L}{\partial y} = 0$$

so-called Euler-Lagrange equations whose solutions are the paths of the semispray ξ on Three-Dimensional almost Para- Kenmotsu manifolds space form $(M^3, \phi. \xi, \eta, g)$. Finally one may say that the triple $(M^3, \phi. \xi, \eta, g)$ is mechanical system on Three-Dimensional almost Para- Kenmotsu manifolds $(M^3, \phi. \xi, \eta, g)$ Therefore we say

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