Initial-Oblique Derivative Boundary Value Problem for Nonlinear Parabolic Equations of Second Order

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Abstract: In this article, we discuss that an initial-oblique derivative boundary value problem for nonlinear uniformly parabolic complex equation of second order

\[ A_0 u_{zz} - \text{Re} [Qu_{zz} + A_1 u_z] - \hat{A} u - u_t = A_3 + G(z, t, u, u_z) \text{ in } G, \]

In a multiply connected domain, the above boundary value problem will be called problem O. If the above complex equation satisfies the conditions similar to Condition C and (1.12), and the boundary conditions satisfy the conditions similar to (1.4)-(1.7) and (1.11) below, then we can obtain some solvability results of Problem O in G.

Keywords: Initial-oblique derivative problem, nonlinear parabolic complex equations, multiply connected domains

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1. FORMULATION OF INITIAL-OBLIQUE DERIVATIVE PROBLEMS FOR SECOND ORDER PARABOLIC COMPLEX EQUATIONS

Let \( D \) be an \((N+1)\)-connected bounded domain in the \( z = x + iy \) plane \( \mathbb{C} \) with the boundary \( \Gamma = \sum_{j=0}^{N} \Gamma_j \subset \mathbb{C} \) without loss of generality, we may consider that \( D \) is a circular domain in \(|z| < 1\) with the boundary \( \Gamma = \sum_{j=0}^{N} \Gamma_j \) where \( \Gamma_j = \{ z \in \mathbb{C}, |z - z_j| = \gamma_j \}, j = 0, 1, ..., N \), \( \Gamma_0 = \Gamma_{N+1} = \{ z \in \mathbb{C}, z = 0 \} \) and \( z = 0 \in D \). Denote \( G = D \times \bar{\mathbb{C}} \), in which \( I = \{ 0 < t \leq T \} \). Here \( T \) is a positive constant, and \( \partial G = \partial G_1 \cup \partial G_2 \) is the parabolic boundary of \( G \), where \( \partial G_1, \partial G_2 \) are the bottom \( \{ z \in D, t = 0 \} \) and the lateral boundary \( \{ z \in \Gamma, t = 1 \} \) of the domain \( G \) respectively.

We consider the nonlinear nondivergent parabolic equation of second order

\[ \Phi(x, y, t, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) - u_t = 0 \text{ in } G, \]

where \( \Phi \) is a real-valued function of \( x, y, t ( \in \mathbb{R} ) \), \( u, u_x, u_y, u_{xx}, u_{xy}, u_{yy} ( \in \mathbb{R}) \). Under certain conditions, the equation (1.1) can be reduced to the complex form

\[ A_0 u_{zz} - \text{Re} [Qu_{zz} + A_1 u_z] - \hat{A} u - u_t = A_3, \]

where \( z = x + iy, \Phi = \Psi(z, t, u, u_z, u_{zz}, u_{zz}), \) and
\[ A_0 = \int_0^1 \Psi_{u_\tau}(z, t, u, u_z, T_{u_\tau z}, T_{u_\tau z}) dT = A_0(z, t, u, u_z, u_{zz}), \]

\[ Q = -2 \int_0^1 \Psi_{u_\tau}(z, t, u, u_z, T_{u_\tau z}, T_{u_\tau z}) dT = Q(z, t, u, u_z, u_{zz}), \]

\[ A_1 = -2 \int_0^1 \Psi_{u_z}(z, t, u, T_{u_z}, 0, 0) dT = A_1(z, t, u, u_z), \]

\[ A_2 = -\int_0^1 \Psi_{u_z}(z, t, u, 0, 0, 0) dT = A_2(z, t, u) + |u|^\sigma, \]

\[ A_3 = -\Psi(z, t, 0, 0, 0, 0) = A_3(z, t), \]

Where \( \sigma \) is a positive constant (see [4]).

Suppose that the equation (1.2) satisfies the following conditions, namely

**Condition C.** (1) \( A_0(z, t, u, u_z, u_{zz}), Q(z, t, u, u_z, u_{zz}), A_1(z, t, u, u_z) \)

\( A_2(z, t, u), A_3(z, t) \) are measurable for any continuously differentiable function \( u(z, t) \in C^{1,0}(G) \) and measurable functions \( u_{zz}, u_{zz} \in L_2(G^*) \) and satisfy the conditions

\[ 0 < \delta \leq A_0 \leq \delta^{-1} \]

\[ |A_j| \leq k_0, j = 1, 2, L_p[A_3, G] \leq k_1, p > 4, \]

where \( G^* \) is any closed subset in the domain \( G \).

(2) The above functions with respect to \( u \in R, u_t \in C \) are continuous for almost every point \((z, t) \in G \)

(3) For almost every point \((z, t) \in G \) and \( u \in R, u_z, U^j \in C, V^j \in R, j = 1, 2, \)

There is

\[ \Psi(z, t, u, u_z, U^1, V^1) - \Psi(z, t, u, u_z, U^2, V^2) \]

\[ = \tilde{A}_0(V^1 - V^2) - \text{Re}[\tilde{Q}(U^1 - U^2)], \delta \leq \tilde{A}_0 \leq \delta^{-1}, \]

\[ \sup_{G} \left( \tilde{A}_0^2 + |\tilde{Q}|^2 \right) \inf_{\tilde{A}_0} \tilde{G} \leq q < 4/3, \]

In (1.4)-(1.7), \( \delta > 0, q (\geq 1), k_0, k_1, p (> 4) \) are non-negative constants. For instance the nonlinear parabolic complex equation

\[ U_{zz} = \tilde{G}(z, t, u, u_z, u_{zz}) + (1 + |u|^4)u + u_0 \]

\[ \tilde{G}(z, t, u, u_z, u_{zz}) = \tilde{A}_0 \tilde{G} / |u_{zz}| \text{ for } |u_{zz}| > 1. \]

satisfies Condition C. In this article, the notations are the same as in References [1-8].

Now we explain the derivation of 3/4 in the condition (1.7). Let \( A = r \inf_{G} \tilde{A}_0 > 0, \text{thus } \inf_{G} \tilde{A}^2 = \)

\[ \inf_{G} \tilde{A}_0^2 / A = \inf_{G} \tilde{A}_0^2 / (r \inf_{G} \tilde{A}_0^2) = 1/r. \]

By the requirement below, we need the inequality

\[ \eta = \sup_{G} [(\tilde{A}_0 - 1)^2 + |\tilde{Q}|^2] < 1/4, \text{ i.e. } \sup_{G} [\tilde{A}_0^2 + |\tilde{Q}|^2 - 2 \tilde{A}_0] < 1/4 - 1, \]
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So it is sufficient that
\[ \sup_{G} \frac{|A_3^2 + |Q|^2|}{r^2} G A_3^2} < 2 \cdot \frac{3}{4} \text{ i.e. } \sup_{G} \frac{|A_3^2 + |Q|^2|}{\inf_{G} A_3^2} < 2 r \cdot \frac{3}{4} r^2 = f(r). \]

We can find the maximum of the function \( f(r) = 2r - (3r^2)/4 \) on \((0, \infty)\), due to \( f'(r) = 2 - (3/2) = 0 \).
It is easy to see that \( f(r) \) takes its maximum on \((0, \infty)\) at the point \( r = \frac{4}{3} \), and then \( f(\frac{4}{3}) = 2(\frac{4}{3}) - (\frac{3}{4})(\frac{4}{3})^2 = \frac{4}{3} \), leading to the inequality (1.7). (see \( \left| 2, \frac{4}{3} \right| \))

In this article, we mainly discuss the nonlinear parabolic equation of second order

\[ A_0 u_{zz} \text{Re}[Qu_z + A_1 u_t] - \bar{A}_2 u - u_t = A_1 + F(z, t, u, u_z). \] (1.8)

Satisfying Condition \( C' \), in which the coefficients \( A_j (j = 0, 1, 2, 3), Q \) of equation (1.8) satisfy the conditions (1.4)–(1.7) and \( F(z, t, u, u_z) \) satisfies the condition:

\[ (4) \left| F(z, u, u_z) \right| \leq B_1(z) \left| u_z \right|^4 + B_2(z) \left| u \right|^T, \quad \left| B_j \right| \leq k_0, j = 1, 2, \] (1.9)

for positive constants \( \eta, T, K_0 \). We can see that \( F(z, t, u, u_z) \) implies the nonlinear items.

**Problem O.** The so-called initial-oblique derivative boundary value problem for the equation (1.8) is to find a continuous solution \( u(z, t) \in C^{1,0}(\bar{G}) \) of (1.8) in \( \bar{G} \) satisfying the initial-boundary conditions

\[
\left\{ \begin{array}{l}
\quad u(z, 0) = g(z) \text{ on } \partial G_1 = D, \\
\quad \frac{\partial u}{\partial v} + b_1(z, t)u = b_2(z, t) \text{ on } \partial G_2 \text{ i.e.} \\
\quad 2\text{Re}[\bar{\lambda}(z, t)u_z] + b_1(z, t)u = b_2(z, t) \text{ on } \partial G_2,
\end{array} \right. \] (1.10)

Where \( v \) is the unit vector at every point on \( \partial G_2 \). There is no harm in assuming that \( v \) is parallel to the plane \( t = 0 \). In addition, \( g(z), b_j(z, t)(j = 1, 2) \) and \( \lambda(z, t) = \cos(v, x) - i \cos(v, y) \) are known functions satisfying the conditions

\[
\left\{ \begin{array}{l}
C_2^0 \left[ g, \partial G_1 \right] \leq k_2 \frac{\partial g}{\partial v} + b_1(z, 0)g = b_2(z, 0) \text{on } \partial G_1 \times \{ t = 0 \}, \\
C_{a, a/2}^{0, 0} [\eta, \partial G_2] = C_{a, a/2}^{0, 0} [\eta, \partial G_2] + C_{a, a/2}^{0, 0} [\eta_z, \partial G_2] \leq k_0, \eta = \{ b_1, \lambda \}, \\
C_{a, a/2}^{2, 1} [b_2, \partial G_2] \leq k_3, b_1(z, t) \geq 0, \cos(v, n) > 0 \text{ on } \partial G_2,
\end{array} \right. \] (1.11)

In which \( n \) is the unit outward normal vector at every point on \( \partial G_2 \). \( a(1/2 \leq a < 1), k_0, k_2, k_3 \) are non-negative constants. The above initial-boundary value problem is the initial-oblique derivative boundary value problem (Problem O). In particular, Problem O with the condition \( v = n, a_1(z, t) = 1, a_2(z, t) = 0 \) on \( \partial G_2 \) is the so-called initial-Neumann boundary value Problem, which will be called Problem N. Problem O for (1.2) with \( A_3(z, t) = 0 \) and \( g(z) = 0, b_2(z, t) = 0 \) is called Problem O_0.
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In order to discuss the uniqueness of solutions of Problem O for the equation (1.2), we add the condition: For any \( u^i \in \mathbb{R}, u^j \) (\( i = 1, 2 \)), \( U \in \mathbb{C}; V \in \mathbb{R} \), there is

\[
\Psi(z, t, u^1, u^2, U, V) - \Psi(z, t, u^3, u^2, U, V) = \hat{A}_0(u_1 - u_2)_{zz} - \text{Re}(\hat{Q}u_{zz} + \hat{A}_1(u^1 - u^3) + \hat{A}_2(u^1 - u^3)) \text{ on } \partial G_2
\]

(1.12)

Where \( \hat{A}_0, \hat{Q} \) satisfy (1.7) and \( \hat{A}_j(j = 1, 2) \) satisfy

\[
| \hat{A}_j | < \infty \text{ in } \mathbb{R}, j = 1, 2.
\]

(1.13)

**Theorem 1.1.** Suppose that the equation (1.2) satisfies Condition C and (1.12). Then the solution \( u(z, t) \) of Problem O for (1.2) is unique. Moreover the homogenous Problem O (Problem O_0) of equation (1.2) with \( A_3 = 0 \) only has the trivial solution \( E \).

**Proof.** Let \( u_j (j = 1, 2) \) be two solutions of Problem O for (1.2). It is easy to see that \( u = u_1(z, t) - u_2(z, t) \) is a solution of the following initial-boundary value problem

\[
\begin{aligned}
\hat{A}_0 u_{zz} - \text{Re}(\hat{Q}u_z + \hat{A}_2u_z) - \hat{A}_3u - u_i = 0 \text{ in } G, \\
\frac{\partial u}{\partial n} + b_1(z, t)u = 0 \text{ on } \partial G_2
\end{aligned}
\]

(1.14)

(1.15)

Where

\[
\begin{aligned}
\hat{A}_0 &= \int_0^1 \psi_s(z, t, v, p, q, s) dT, s = u_{zz} + T(u_1 - u_2)_{zz}, q = u_{zz} + T(u_1 - u_2)_{zz}, \\
\hat{Q} &= -2 \int_0^1 \psi_q(z, t, v, p, q, s) dT, p = u_{zz} + T(u_1 - u_2)_{zz}, v = u_{zz} + T(u_1 - u_2), \\
\hat{A}_1 &= -2 \int_0^1 \psi_p(z, t, v, p, q, s) dT, \hat{A}_2 = \int_0^1 \psi_v(z, t, v, p, q, s) dT
\end{aligned}
\]

(1.16)

Introducing a transformation \( v = v(z, t) = u e^{Bt} \), where \( B \) is an undetermined real constant, the complex equation (1.14) and the initial-boundary condition (1.15) can be reduced to the form

\[
\hat{A}_0 v_{zz} - \text{Re}(\hat{Q}v_z + \hat{A}_1v_z) - (\hat{A}_2 + B)v - v_i = 0
\]

(1.17)

(1.18)

Let the above equation be multiplied by \( v \), thus an equation of \( v^2 \):

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\[
\frac{1}{2} \{ \hat{A}_0(v^2)_{zz} - \text{Re}(\hat{Q}(v^2)_z) \}
\]

(1.19)

Can be obtained If the maximum of \( v^2 \) occurs at an inner point \( P_0 \in G \) with \( |v(P_0)|^2 \neq 0 \), then in a neighborhood of \( P_0 \), the right hand side of (1.19) \( \geq [B - k_0]v^2 \). Moreover, we choose the constant \( B \) such that \( B > k_0 \). By using the maximum principle (see [3,4]), the function \( v^2 \) cannot take the positive maximum in \( G \). If \( v^2 \) takes the positive maximum at a point \( P_0 \in \partial G_2 \), then we have
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\[
\left. \frac{\partial u}{\partial v} + b_t(z, t) v^2 \right|_{p=p_0} > 0.
\]  

(1.20)

This contradicts (1.18). Hence we derive that \( u = 0 \), i.e: \( u_1 - u_2 = 0 \) in \( \tilde{G} \). Similarly we can prove the other part in this theorem.

2. A PRIOR ESTIMATE OF SOLUTIONS OF THE INITIAL-OBLIQUE DERIVATIVE PROBLEM OF SECOND ORDER PARABOLIC COMPLEX EQUATIONS

Theorem 2.1. If the equation (1.2) satisfies condition C, then the solution \( u(z, t) \) of Problem O for (1.2) satisfies the estimate

\[
\hat{C}_{\beta, \beta, 2}[u, \tilde{G}] = C_{\beta, \beta, 2}[u, \tilde{G}] \leq M_1, \quad \| u \| \leq M_2, \quad \omega^2(G) \leq M_2,
\]  

(2.1)

Where \( \beta (0 < \beta \leq \alpha) \), \( k = k(k_0, k_1, k_2, k_3) \). \( M_1 = M_j(\delta, q, p, \beta, k, G) \) \((j = 1, 2)\) are non-negative constants only dependent on \( \delta, q, p, \beta, k, G \).

Proof. We shall prove that the following estimate holds

\[
\hat{C}_{1, 0}[u, \tilde{G}] = C_{1, 0}[u, \tilde{G}] \leq M_3 = M_3 (\delta, q, p, \beta, k, G),
\]  

(2.2)

If (2.2) is not true, then there exists a sequence of parabolic equations

\[
A_0^m u_{zz} + \text{Re}[Q_0^m u_{zz} + A_3^m u_z] - A_3^m u - u_t = A_3^m \text{ in } G,
\]  

(2.3)

and a sequence of initial-boundary conditions

\[
\begin{align*}
\frac{\partial u}{\partial v} + b_t^m u = b_t^m \text{ on } \partial G_2,
\end{align*}
\]  

(2.4)

with \( \{A_0^m\}, \{Q_0^m\}, \{A_1^m\}, \{A_3^m\}, \{A_3^m\} \) in \( G \) satisfying Condition C and \( g^m, b_0^m, b_2^m \) satisfying (1.11), where \( \{A_0^m\}, \{Q_0^m\}, \{A_1^m\}, \{A_3^m\}, \{A_3^m\} \) in \( G \) weakly converge to \( A_0^0, Q_0^0, A_1^0, A_3^0, A_3^0 \) and \( g^0(z), b_0^0(z, t), b_2^0(z, t) \) in \( D, \partial G_2 \) uniformly converge to \( g^0(z), b_0^0(z, t), b_2^0(z, t) \) respectively, and the initial-boundary value problem (2.3)–(2.4) have the solution \( u^m(z, t) \in C_{1, 0}(\tilde{G}) \) \((m = 1, 2, \ldots)\) such that \( \hat{C}_{1, 0}[u^m, \tilde{G}] = H_m \to \infty \) as \( m \to \infty \). There is no harm in assuming that \( H_m \geq \max[k_1, k_2, \ldots, 1] \). Let \( U^m = u^m / H_m \), it is easy to see that \( U^m \) satisfies the complex equation and initial-boundary conditions

\[
\begin{align*}
A_0^m U_{zz} - \text{Re}[Q_0^m U_{zz} + A_3^m U_z] - A_3^m U - U_t = A_3^m / H_m \text{ in } G, \\
U^m(z, 0) = g^m(z) = H_m \text{ on } D, \\
\frac{\partial U^m}{\partial v} + b_t^m U = b_t^m / H_m \text{ on } \partial G_2.
\end{align*}
\]  

(2.5)

We can see that the some coefficients in the above equation and boundary conditions satisfy the condition C and

\[
\left| u^m \right|^{\alpha+1} / H_m \leq l, \quad L_p [A_3^m / H_m, \tilde{G}] \leq l
\]

\([C_1 g^m(z)/H_m, D] \leq l, \quad \left| b_2^m / H_m \right| \leq l
\]
Hence by Theorem 5.3.1, [7], we can obtain the estimates
\[ \hat{c}^{1.0}_{\beta, \beta, \beta/2}[u^m, \tilde{G}] \leq M_\beta, \quad \| u^m \|_{W^{2,1}_2(G)} \leq M_\beta. \tag{2.6} \]

In which \( \beta (0 < \beta \leq \alpha), M_\beta = M_\gamma(\delta, q, p, \beta, k, G)(j = 4, 5) \) are non-negative constants. Thus from \( \{U^m\}, \{U^m\} \) we can select the subsequences \( \{U^{mk}\}, \{U^{mk}\} \) such that they uniformly converge to \( U^0, U^0 \) in \( \tilde{G} \) and \( \{U^{mk}_{zz}\}, \{U^{mk}_{zt}\} \) weakly converge to \( U^0_{zz}, U^0_{zt}, U^0_t \) in \( G \) respectively, and \( U^0 \) is a solution of the following initial - boundary value problem

\[
\begin{align*}
A^0_{zz} U^0_{zz} - \text{Re}[Q U^0_{zz} + A^0_{zt} U^0_{zt} + \hat{A}^0 U^0] - U^0_t &= 0 \text{ in } D, \\
\frac{\partial U^0}{\partial v} + b^0_t U^0 &= 0 \text{ on } \partial G.
\end{align*}
\tag{2.7}
\]

From Theorem 1.1, we see that \( U^0 = 0 \). However, from \( \hat{c}^{1.0}[U^m, \tilde{G}] = 1 \), there exists a point \( (z^*, t^*) \in \tilde{G} \), such that \( |U^0(z^*, t^*)| > 0 \). This contradiction shows that the estimate (2.2) is true. Moreover, by using the method from (2.2) to (2.6), two estimates in (2.1) can be derived.

**Theorem 2.2.** Suppose that Condition \( C' \) holds. Then any solution \( u(z, t) \) of Problem \( 0 \) for (1.8) satisfies the estimates

\[ \hat{c}^{1.0}_{\beta, \beta, \beta/2}[u, \tilde{G}] = C^{1.0}_{\beta, \beta, \beta/2}[u, \tilde{G}] \leq M_\beta k', \quad \| u \|_{W^{2,1}_2(G)} \leq M_j k', \tag{2.8} \]

Where \( \beta (0 < \beta \leq \alpha), k' = k_1 + k_2 + k_3 + k_0(\| u \|^{\eta} + \| u \|^{T}), M_j = M_j(\delta, q, p, \beta, k_0, G)(j = 6, 7) \) are non-negative constants.

**Proof.** If \( k' = 0 \), i.e. \( k_0 = k_1 = k_2 = k_3 = 0 \), from Theorem 1.1, it follows that \( u(z) = 0 \) in \( \tilde{G} \). If \( k' > 0 \), it is easy to see that \( U(z) = u(z)/k' \) satisfies the complex equation and boundary conditions

\[
A^0_{zz} \text{Re}[Q U_{zz} + A^0_{zt} U_{zt} - A^1 u_t] = [A^1 + F(z, t, u, u_z)]/k', \tag{2.9}
\]

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\[
\begin{align*}
\frac{d u}{d v} + b_1(z, t) u &= \frac{b_2(z, t)}{k'}(z, t) \in \partial G, \\
\| u \|_{L^p[G/k']} &\leq 1, \quad C^1_{\alpha}[g/k', D] \leq 1, \quad C^{1.0}_{\alpha, \alpha/2}[b_2/k', \partial G] \leq 1.
\end{align*}
\tag{2.10}
\]

Noting that

\[ L_p[A^1(z, t)/k', \tilde{G}] \leq 1, \quad C^1_{\alpha}[g/k', D] \leq 1, \quad C^{1.0}_{\alpha, \alpha/2}[b_2/k', \partial G] \leq 1. \]

And according to the proof of Theorem 2.1, we have

\[ \hat{c}^{1.0}_{\beta, \beta, \beta/2}[U, \tilde{G}] \leq M_\beta, \quad \| U \|_{W^{2,1}_2(G)} \leq M_\beta, \]

From the above estimates, it immediately follows that two estimates in (2.8) hold.

### 3. Solvability of the Initial-Oblique Derivative Problem of Second Order Parabolic Complex Equations

We consider the complex equation (1.8) namely the equation

\[
A^0_{zz} \text{Re}[Q U_{zz}] - u_t = f(z, t, u, u_z), f(z, t, u, u_z) =
\]
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\[ \text{Re} \left[ Qu_{zz} + A_1 u_z \right] + \hat{A}_2 u + A_3 + F(z, t, u, u_z) \text{ in } G, \]  \hspace{1cm} (3.1)

in which \( A_0 = A_0(z, t, u, u_z, u_{zz}), Q = Q(z, t, u, u_z, u_{zz}), A_1 = A_1(z, t, u, u_z), \hat{A}_2 = A_2(z, t, u) + |u|^\alpha \)
\( A_3 = A_3(z, t). \)

**Theorem 3.1.** Suppose that equation (1.8) satisfies Condition C', and (1.12).

1. When \( 0 < \eta, T < 1, \) Problem O for (1.8) has a solution \( u(z, t) \in C^{1,0}(\bar{G}). \)
2. When \( \min (\eta, T) > 1, \) Problem O for (1.8) has a solution \( u(z, t) \in C^{1,0}(\bar{G}), \) provided that

\[ M_8 = L_p[A_3, \bar{G}] + C^2_{q, \hat{A}}[g, \bar{D}] + C^2_{q, \hat{A}}[b_2, \partial G_2] \]  \hspace{1cm} (3.2)

is small enough.
3. When \( F(z, t, u, u_z) \in (1.8) \) possesses the form

\[ F(z, t, u, u_z) = \text{Re} B_0 u_z + B_2 u t^2 \text{ in } D \]  \hspace{1cm} (3.3)

In which \( 0 < T < \infty, L_p[B_j, \bar{D}] \leq k_0 (< \infty, p > 4, j = 1, 2) \) with a positive constant \( k_0, \) if \( T < 1, \) and if \( T > 1 \) and \( M_8 \) in (3.2) is small enough, then (1.8) has a solution \( u(z, t) \in C^{1,0}(\bar{G}). \)

**Proof.** (1) Consider the algebraic equation for \( t \)

\[ M_8[k_1 + k_0(t^9 + t^6)] + k_2 + k_3 = t. \]  \hspace{1cm} (3.4)

Because \( 0 < \eta, T < 1, \) the above equation has a solution \( t = M_9 > 0, \) which is also the maximum of \( t \) in \( (0, +\infty). \) Now, we introduce a closed, bounded and convex subset \( B \) of the Banach space \( C^{1,0}(\bar{G}); \) whose elements are of the form \( u(z) \) satisfying the condition

\[ C^{1,0}[u(z)]_{\alpha+1}(\bar{G}) \]  \hspace{1cm} \( \leq M_9. \)  \hspace{1cm} (3.5)

We choose an arbitrary function \( u(z) \in B \) and substitute it into the proper positions in the following equation and initial-boundary conditions (Problem \( O^h \)) with the parameter \( h \in [0, 1] \)

\[ \begin{aligned}
A_0 u_{zz} - \text{Re}[u_{zz}] - u_t - h f(z, t, u, u_z) & = A(z, t), (z, t) \in G, \\
A_1(0) & = g(z), z \in D, \\
\frac{\partial u}{\partial v} + h b_1(z, t) u & = b(z, t), (z, t) \in \partial G_2
\end{aligned} \]  \hspace{1cm} (3.6)

where \( A(z, t) \) are any measurable functions with the condition \( A(z, t) \in L_p(\bar{G}), p > 4, \) and \( b(z, t) \) is a continuously differentiable function with the condition \( b(z, t) \in C^{1,0}_{\alpha, \beta/2}(\partial G_2). \) When \( h = 0, \) according to Theorem 4.3,[4], we see that there exists a solution \( u_0(z, t) \in B = \hat{C}^{1,0}_{\alpha, \beta/2}(\bar{G}) \cap W^{2,1}_2(G) \) of Problem \( O^0. \) Suppose that when \( h = h_0, (0 \leq h_0 < 1), \) Problem \( O^{h_0} \) for (3.6) is solvable. We shall prove that there exists a positive constant \( \epsilon \) independent of \( h_0, \) such that for any \( h \in E = \{|h - h_0| \leq \epsilon, 0 \leq h \leq 1\}, \) Problem \( O^h \) for (3.6) possesses a solution \( u(z, t) \in B; \) Let the above problem be rewritten in the form
We arbitrarily choose a function \( u_0(z, t) \in B \) and substitute it into the position of \( u \) on the right hand side of (3.7). It is easily seen that

\[
(h-h_0) f(z, t, u_0) + A(z, t) \in L^p(\bar{G})
\]

By the hypothesis of \( h_0 \), there exists a solution \( u_1(z, t) \in B \) of Problem \( O_h \) corresponding to

\[
\left\{
\begin{array}{ll}
A_0 u_{zz} - \text{Re}[Qu_{zz}] - u_t - h_0 f(z, t, u, u_z) = (h-h_0)f(z, t, u, u_z) + A(z, t) & \text{in } G \\
\frac{\partial u}{\partial v} + h_0 b_1 u = (h_0 - h)b_1 + b(z, t) & \text{on } \partial G_2
\end{array}
\right.
\]

By using the successive iteration, we obtain a sequence of solutions \( u^m(z, t) \) \((m = 1, 2, \ldots) \in B \) of Problem \( O^h \), which satisfy

\[
\left\{
\begin{array}{ll}
A_0 u_{zz} - \text{Re}[Qu_{zz}] - u_t - h_0 f(z, t, u, u_z) = (h-h_0)f(z, t, u, u_z) + A(z, t) & \text{in } G \\
\frac{\partial u^m}{\partial v} + h_0 b_1 u^m = (h_0 - h)b_1 u^0 + b(z, t) & \text{on } \partial G_2
\end{array}
\right.
\]

According to the way in the proof of Theorem 2.2, we can obtain

\[
C^{1,0}[u^m, \mathcal{G}] = ||u^m|| \leq ||h-h_0||M_{10}C^{1,0}[u^m, \mathcal{G}]
\]

where \( M_{10} = M_{10}(\delta, q, p, \beta, k, G) \geq 0 \). Setting \( \epsilon = 1/2(M_{10} + 1) \), we have

\[
||u^m|| \leq \frac{1}{2} ||u^m|| \text{ for } h \in E.
\]

Hence when \( n \geq m > N + 2(> 2) \), there are

\[
||u^{m+1} - u^m|| = 2^{N} ||u^1 - u^0||.
\]

This shows that \( ||u^m - u^n|| \to 0 \) as \( n, m \to \infty \). By the completeness of the Banach space \( B \), there exists \( u^* \in B \) such that \( ||u^* - u^m|| \to 0 \) as \( n \to \infty \) and \( u^* \) is the solution of Problem \( O^h \) with \( h \in E \). Thus from the solvability of Problem \( O^0 \), we can derive the solvability of Problem \( O^1 \), in particular Problem \( O_1^1 \) with \( A = 0 \) and \( b(z, t) = 0 \), i.e. Problem \( O \) for (3.1) has a solution. This completes the proof.

(2) For the case \( \min(\eta, T) < 1 \), due to \( M_8 \) in (3.2) is small enough, from
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\[ M_0[k_1 + k_0(t^p + t^q) + k_2 + k_3] = t, \]

a solution \( t = M_{11} > 0 \) can be solved, which is also a maximum. Now we consider a subset \( B^* \) in the Banach space \( C^1(\overline{\Omega}) \), i.e.

\[ B^* = \{ u(z) \mid C^{1,0}[u, \overline{\Omega}] \leq M_{11} \} \]

and apply a similar method as before. We can prove that there exists a solution \( u(z) \in B^* = C^{1,0}(\overline{\Omega}) \) of Problem O for (1.8) with the constant \( \min(\eta, T) > 1 \).

(3) By using the similar method as in proofs of (1) and (2), we can verify the solvability of problem O for (1.8) with the conditions \( 0 < T < 1 \) and \( 1 < T < \infty \) as in (3) of the theorem.

REFERENCES