Singular Modified Riemann-Hilbert Problems for Nonlinear Elliptic Complex Equations of First Order

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Abstract: In [1], the author first proposed a well-posedness of singular Riemann-Hilbert boundary value problem for generalized analytic functions in multiply con-nected domains, and the well posedness allows that the solutions of the modified problem possess some poles in N + 1-connected domain D. In [3], the author proposed another well-posedness of the Riemann-Hilbert boundary value problem with continuous solutions for nonlinear elliptic complex equations of first order, in particular the well-posedness includes the well-posedness of the singular case of 0 < K < N. Recently, the authors of this paper proposes three kinds of new well-posedness of singular Riemann-Hilbert boundary value problem for nonlinear elliptic complex equations of first order in multiply connected domains. We shall prove the existence of solutions for these boundary value problems.

Keywords: Singular modified Riemann-Hilbert problem, elliptic complex equations of first order, three kinds of well posedness with pole points, the existence of solutions.

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1. FORMULATION OF SINGULAR MODIFIED RIEMANN-HILBERT BOUNDARY VALUE PROBLEMS FOR ELLIPTIC COMPLEX EQUATIONS OF FIRST ORDER

First of all, we introduce the nonlinear elliptic equations of first order

\[ \begin{align*}
F(z, w, w_z) &= Q_1 w_z + Q_2 \bar{w}_z + A_1 w + A_2 \bar{w} + A_3, \\
F_j(z, w, w_z) &= 1, 2, A_j(\overline{z}, w) = 1, 2, 3. 
\end{align*} \]

In a bounded N + 1 (N ≥ 1)-connected domain D, which is the complex form of the real nonlinear elliptic system of first order equations

\[ \Phi_j(x, y, u, v, u_x, v_x, v_y) = 0, \quad j = 1, 2 \]

Under certain conditions (see Theorem 1.2, Chapter I, [4]). There is not harm in assuming that D is an N + 1 (N ≥ 1)-connected circular domain in \(|z| < 1\) bounded by the \((N + 1)\)-circles \(\Gamma_j : |z - z_j| = r_j, j = 0, 1, ..., N \) and \(\Gamma_0 = \Gamma_{N+1} : |z| = 1, z = 0 \in D\). In this article, the notations are as the same in References [3-13]. Suppose that the complex equation (1.1) satisfies the following conditions, namely

**Condition C.**
1) \(Q_j(z, w, U) (j = 1, 2)\), \(A_j(z, w)(j = 1, 2, 3)\) are measurable in \(z \in D\) for all continuous functions \(w(z)\) in \(D(0)\) and all measurable functions \(U(z) \in L_p(D)\), and satisfy

\[ \begin{align*}
L_p[A_j, \overline{D}] &\leq k_0, j = 1, 2, L_p[A_3, \overline{D}] \leq k_1, \\
\end{align*} \]

where \(p, p_0 \leq p, k_0, k_1\) are non-negative constants.

2) The above functions are continuous in \(w \in C\) for almost every point \(z \in D\), \(U \in C\), and \(A_j = 0 (j = 1, 2, 3)\) for \(z \notin CD\).

3) The complex equation (1.1) satisfies the uniform ellipticity condition, i.e. for any \(U_1, U_2 \in C\),
the following inequality in almost every point $z \in D$ holds:

$$\left| F(z, w, U_1) - F(z, w, U_2) \right| \leq q_0 \left| U_1 - U_2 \right|,$$

(1.3)

In which $q_0 (\leq 1)$ is a non-negative constant.

It is well known that a generalized analytic function in a domain $D$ is a continuous solution of the complex equation

$$w_z = A(z)w + B(z)\overline{w}, \quad z \in D,$$

(1.4)

Where $z = x + iy, w_z = [w_x + i w_y]i2, A(z), B(z) \in L_p(\overline{D}) (p > 2)$; the conditions will be called Condition $C_0$. Obviously the complex equation (1.4) is a special case of (1.1).

Now we first formulate the new singular Riemann-Hilbert problem with the non-negative index for equation (1.1) as follows.

**Problem B**. The singular modified Riemann-Hilbert boundary value problem for (1.1) is to find a continuous solution $w(z)$ in $\overline{D}$ with the pole point of $n$ order at the point $z = 0 (\in D)$ satisfying the boundary condition:

$$\text{Re} \left[ \lambda(z) w(z) \right] = r(z) + h(z), \quad z \in \Gamma,$$

(1.5)

Where $\lambda(z), r(z)$ satisfy the conditions

$$C_a[\lambda(z), \Gamma] \leq k_a, \quad C_a[r(z), \Gamma] \leq k_2, \text{ in which}$$

(1.6)

$$\lambda(z) = a(z) + ib(z) \quad \text{on} \quad \Gamma, \quad a \left( 1/2 < a < 1 \right) \quad \text{is a positive constant. The index} \quad K \quad \text{of Problems B}_1 \quad \text{is defined by:}$$

$$K = K_0 + K_1 + \ldots \ldots + K_N = \sum_{j=0}^{N^j} \frac{1}{2\pi} \Delta_{1j} \text{arg} \lambda(z) \geq 0,$$

(1.7)

The partial indexes $K_j = \Delta_{1j} \text{arg} \lambda(z)/2\pi (j = 0, 1, \ldots, N)$ of $\lambda(z)$ are integers and

$$h(z) = \left\{ \begin{array}{ll}
0, & z \in \Gamma_0, \\
\frac{1}{b_j}, & z \in \Gamma_1, j \in 1, \ldots, N,
\end{array} \right.$$  

(1.8)

$b_j (j = 1, \ldots, N)$ are unknown real constants to be determined appropriately. Moreover we assume that the solution $w(z)$ satisfies the following point conditions

$$\text{Im} \left[ \lambda(a_j) w(a_j) \right] = b_j, \quad j \in J = \{1, \ldots, 2K + 1\},$$

(1.9)

in which $a_j \in \Gamma_0 (j = 1, \ldots, 2K + 1)$ are distinct fixed points, and $b_j (j \in J)$ are all real constants satisfying the conditions

$$|b_j| \leq k_b, \quad j \in J,$$

(1.10)

herein $k_b$ is a non-negative constant. Problem B with $A_z(z, w) = 0$ in $D, r(z) = 0$ on $\Gamma$ and $b_j (j \in J)$ is called Problem B$_2$.

Next we shall introduce the other two kinds of well-posedness of new singular Riemann-Hilbert boundary value problem for the equation (1.1) as follows.

**Problem B**. To find a continuous solution $w(z)$ of the equation (1.1) in $D\setminus\{0\}$ satisfying the modified boundary conditions

$$\text{Re} \left[ \lambda(z) w(z) \right] = r(z) + h(z), \quad z \in \Gamma,$$

$$\text{Im} \left[ \lambda(a_j) w(a_j) \right] = b_j, \quad j \in J = \{1, \ldots, 2K\},$$

$$w(0) = \infty, \quad w(a) = 0, \quad w(1) = 1,$$

(1.11)

where $a (\in D)$ is a point, and $\lambda(z), r(z), h(z)$ are the same as in (1.5)-(1.6), and $a_j (\neq 1) \in \Gamma_0$ $(j = 1, \ldots, 2K)$ are distinct fixed points, $b_j (j \in J)$ are all real constants satisfying the conditions

$$|b_j| \leq k_b, \quad j \in J,$$

(1.12)

herein $k_b$ is a non-negative constant.

**Problem B**. To find a continuous solution $w(z)$ of the equation (1.1) in $D\setminus\{0\}$ with the pole point...
of \( n > 0 \) order at \( z = 0 \) and the zero point of \( m (0 < m < n) \) order at \( z = a \) \((c, D, a \neq 0)\) satisfying the modified boundary conditions

\[
\text{Re} \left[ \lambda(z)w(z) \right] = r(z) + h(z), z \in \Gamma,
\]

\[
\text{Im} \left[ \lambda(a)w(a) \right] = b_j, j \in J = \{1, \ldots, 2K + 1\},
\]

in which \( n, m \) \((< n)\) are positive integers and \( \lambda(z), r(z), h(z) \) are the same as in (1.5)-(1.6), and \( a_j \in \Gamma_0 \) \((j \in J = 1, \ldots, 2K + 1)\) are distinct fixed points, \( b_j \) \((j \in J)\) are all real constants satisfying the condition

\[
|b_j| \leq k_j, j \in J
\]

with the constant \( k_j \).

In order to prove the solvability of Problem B\(_1\) for the complex equation (1.1), we need to give a representation theorem for Problem B\(_1\).

**Theorem 1.1.** Suppose that the complex equation (1.1) satisfies Condition C, and \( w(z) \) is a solution of Problem B\(_1\) for (1.1). Then \( w(z) \) is represented by

\[
w(z) = [\Phi(z)] + \Psi(z)e^{\Phi(z)},
\]

where \( \zeta(z) \) is a homeomorphism in \( \tilde{D} \), which quasiconformally maps \( D \) onto the \( N + 1 \)-connected circular domain \( G \) with boundary \( L = \zeta(\Gamma) \) in \( \{|z| < 1\} \), such that \( \zeta(0) = 0 \) and \( \zeta(1) = 1 \), \( \Phi(z) \) is an analytic function in \( G \), \( \Psi(z), \Phi(z) \). \( \zeta(z) \) and its inverse function \( z(\zeta) \) satisfy the following estimates

\[
C_\Phi \left[ \Phi, \tilde{D} \right] \leq k_\Phi, C_\Psi \left[ \Psi, \tilde{D} \right] \leq k_\Psi,
\]

\[
L_{00} \left[ |\Phi_z| + |\Psi_z|, \tilde{D} \right] \leq k_{10}, L_{00} \left[ |\Phi_z| + |\Psi_z|, \tilde{D} \right] \leq k_{10},
\]

\[
C_{\Phi}[\zeta, \tilde{D}] \leq k_{\Phi}, C_{\Psi}[\zeta, \tilde{D}] \leq k_{\Psi},
\]

in which \( \chi(z) \) is as stated in (1.21) below, \( \beta = \min(a, 1 - 2p_0), p_0(2 < p_0 \leq p), k_j = k_j(q_0, p_0, \beta, k_0, k_1, D) \) \((j = 4, 5)\) are non-negative constants dependent on \( q_0, p_0, \beta, k_0, k_1, D \). Moreover, the function \( \Phi(z) \) satisfies the estimate

\[
C_\Phi[\zeta, \tilde{D}] \leq M_1 = M_1(q_0, p_0, \beta, k, D) < \infty,
\]

and \( T \leq \min(a, 1 - 2p_0 - k = k(k_0, k_1, k_2, k_3) \), and \( M_1 \) is a non-negative constant dependent on \( q_0, p_0, \beta, k, D \). Here we mention that the pole of \( n \) order at \( z = 0 \) of \( w(z) \) is denoted the pole of \( n \) order of the function \( \Phi(z) \) at \( z(\zeta) = 0 \).

**Proof.** We substitute the solution \( w(z) \) of Problem B\(_1\) into the coefficients of equation (1.1) and consider the following system

\[
\Phi_z = Q\Phi_z + A, A = A_1 + A_2 + \frac{\gamma}{w} \text{ for } w(z) \neq 0, 0 \text{ for } w(z) = 0 \text{ or } z \notin D,
\]

\[
\Psi_z = Q\Psi_z + A_3 e^{-\Phi(z)}, Q = Q_0 + Q_3 e^{-\Phi}, \text{ for } w(z) \neq 0, 0 \text{ for } w(z) = 0 \text{ or } z \notin D,
\]

\[
w(z) = Qw(z), W(z) = \Phi(z) \text{ in } D.
\]

By using the continuity method and the principle of contracting mapping, we can find the solution

\[
\Psi(z) = T_{\Omega}f = \frac{-1}{\pi} \int_{\Omega} f(z, \zeta) d\sigma, \quad \Phi(z) = T_{\Omega}\zeta \text{ in } \tilde{D},
\]

of (1.20), in which \( f(z), g(z), h(z) \) \( L_{00}(\tilde{D}), 2 < p_0 \leq p \), \( \chi(z) \) is a homeomorphic solution of the third equation in (1.20), \( \Psi(z) \) is a univalent analytic function, which con-formally maps \( E = \chi(D) \) onto the domain \( G \) (see[1,3], and \( \Psi(z) \) is an analytic function in \( G \) such that the function \( \zeta(z) = \Psi(z) \) satisfies \( \zeta(0) = 0 \), \( \zeta(1) = 1 \). We can verify that \( \Psi(z), \Phi(z) \), \( \zeta(z) \) satisfy the estimates (1.16) and (1.17). It remains to prove that \( z = z(\zeta) \) satisfies the estimate in (1.18). In fact, we can find a homeomorphic solution of the last equation in (1.20) in the form \( \chi(z) = z + T_j h \) such that \( [\chi(z)]_z, [\chi(z)]_z \in L_{00}(\tilde{D}) \) (see[1]). By the result on conformal mappings, applying the method of Theorem...
3.2, Chapter V,[4], we can prove that (1.18) is true. It is easy to see that the function \( \Phi_1 \) satisfies the boundary conditions

\[
\text{Re}\left(\overline{\lambda(z)}e^{\phi(z)} \Phi_1(z)\right) = c(z) + h(z) - \text{Re}\left(\overline{\lambda(z)}e^{\phi(z)} \Psi(z)\right), \quad z \in \Gamma
\]

On the basis of the estimates (1.16) and (1.18), and using the methods of Theorems 3.2–3.3, Chapter V, [3], we can prove that \( \Psi_2(z) \) satisfies the estimate (1.19).

2. Unique solvability of Problems \( B_1, B_2, B_3 \) for generalized analytic functions

In this section, we first prove the uniqueness and solvability of Problems \( B_j \) (\( j = 1, 2, 3 \)) for generalized analytic functions.

**Theorem 2.1.** Suppose that equation (1.4) satisfies Condition \( C_0 \). Then the solution of Problem \( B_1 \) are existence and unique

**Proof.** Problem \( B_1 \) for (1.4) can be rewritten as

\[
\text{Re}\left(\overline{\lambda(z)}[1/z^2]W(z)\right) = r(z) + h(z) \text{ in } D,
\]

\[
\text{Im}\left(\overline{\lambda(z)} \right) [1/\alpha_j^2]W(a_j) = b'_j, \quad j \in J = \{1, \ldots, 2(K + n)\},
\]

Where \( W(z) = w(z)/\Psi(z), \quad \Psi(z) = 1/z^2, \quad b'_j (j \in J) \) are real constants with the conditions

\[
|b'_j| < k_4^j < \infty (j \in J).
\]

It is easy to see \( W(z) \) satisfies the complex equation

\[
W_2 = A(z)W + B(z)\Psi(z) / \Psi(z)\overline{W}, \quad z \in D,
\]

The index of \( \lambda(z) \) \( 1/z^2 \) on \( \Gamma \) is equal to \( \lambda + n \) \( > 0 \), the boundary value problem (2.1,2.2) is called Problem \( B_1 \). According to the method in the previous section, we can derive that Problem \( B_1 \) has a unique continuous solution \( W(z) \) in \( \overline{D} \), and then Problem \( B_1 \) for (1.4) is uniquely solvable.

**Theorem 2.2.** Suppose that equation (1.4) satisfies Condition \( C_0 \). Then the solution of Problem \( B_2 \) are existence and unique.

**Proof.** Problem \( B_2 \) for (1.4) can be rewritten as

\[
\text{Re}\left(\overline{\lambda(z)}[(z - a)(1 - a)z]W(z)\right) = r(z) + h(z) \text{ in } D,
\]

\[
(1 - a)(1 - a)W(1) = 1,
\]

Where \( W(z) = w(z)/\Psi(z), \quad \Psi(z) = (z - a)/(1 - a)z \). It is easy to see \( W(z) \) satisfies the complex equation

\[
W_2 = A(z)W + B(z)\Psi(z) / \Psi(z)\overline{W}, \quad z \in D,
\]

the index of \( \lambda(z) \) \((z - a)/(1 - a)z\) on \( \Gamma \) is equal to \( \lambda \), the boundary value problem (2.3,2.4) and the second part of (1.11) is called Problem \( B_2' \), hence according to the result as in Theorem 3.3, Chapter V,[4], f we can derive that Problem \( B_2' \) has a unique continuous solution \( W(z) \) in \( \overline{D} \), and then Problem \( B_2 \) for (1.4) is uniquely solvable.

**Theorem 2.3.** Suppose that equation (1.4) satisfies Condition \( C_0 \). Then the solution of Problem \( B_3 \) is existence and unique.

**Proof.** For problem \( B_3 \) for (1.4) can be rewritten as

\[
\text{Re}\left(\overline{\lambda(z)}[(z - a)^m/z^2]W(z)\right) = r(z) + h(z) \text{ in } D,
\]

\[
\text{Im}\left(\overline{\lambda(z)} \right) [(a_j - a)^m/\alpha_j^2]W(a_j) = b'_j, \quad j \in J = \{1, \ldots, 2(n - m + K)\},
\]

Where \( W(z) = w(z)/\Psi(z), \quad \Psi(z) = (z - a)^m/z^2, \quad b'_j (j \in J) \) are real constants with the conditions

\[
|b'_j| \leq k_4^j < \infty (j \in J).
\]

It is easy to see \( W(z) \) satisfies the complex equation

\[
W_2 = A(z)W + B(z)\Psi(z) / \Psi(z)\overline{W}, \quad z \in D,
\]

That the index of \( \lambda(z) \) \((z - a)^m/z^2\) on \( \Gamma \) is equal to \( \lambda + n - m \) \( > 0 \), the boundary value problem (2.5,2.6) is called Problem \( B_3 \). Moreover we can derive that Problem \( B_3' \) has a unique continuous solution \( W(z) \) in \( \overline{D} \), and then Problem \( B_3 \) for (1.4) is uniquely solvable.
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In the following section, by using Theorem 3.3, Chapter V,[4] we can prove the solvability of Problems $B_j$ \((j = 1, 2, 3)\) for (1.1).

**3. ESTIMATES OF SOLUTIONS AND SOLVABILITY OF PROBLEMS \(B_j\) \((j = 1, 2, 3)\) FOR NONLINEAR ELLIPTIC COMPLEX EQUATIONS IN MULTICYLINDRICAL DOMAINS**

The singular modified Riemann-Hilbert problem (Problem $B_1$) can be transformed into the continuous modified Riemann-Hilbert problem (Problem $B'_1$) as follows.

**Problem $B'_1$.** The modified Riemann-Hilbert boundary value problem for (1.1) is to find a continuous solution $w(z)$ in $\mathbb{D}$ satisfying the boundary condition:

$$\text{Re} \left[ \frac{\lambda(z)}{\xi w(z)} \right] = r(z) + h(z), \ z \in \Gamma, \tag{3.1}$$

Where $\lambda(z), r(z)$ satisfy the conditions

$$C_0[\lambda(z), \Gamma] \leq k_0, \ C_0[r(z), \Gamma] \leq k_2, \tag{3.2}$$

$$\lambda(z) = a(z) + ib(z), \ \text{Re}\lambda(z) = 1 \text{ on } \Gamma, \ \text{and } \alpha (1/2 < \alpha < 1) \text{ is a positive constant.} \text{ The index } K \text{ of Problems } B_1 \text{ is defined as follows:}$$

$$K + 1 = K_0 + K_1 + \ldots + K_N = \sum_{j=0}^{N} \frac{1}{2\pi} \Delta r_j \text{arg}\lambda(z) \geq 0, \tag{3.3}$$

The partial indexes $K_j = \Delta r_j \text{arg}\lambda(z)/2\pi$ of $\lambda(z)$ are integers. And

$$h(z) = \begin{cases} 0, & \text{if } z \in \Gamma, \\ \beta, & \text{if } z \notin \Gamma, \end{cases} \tag{3.4}$$

$h_j (j = 1, \ldots, N)$ are unknown real constants to be determined appropriately. Moreover we assume that the solution $w(z)$ satisfies the following point conditions

$$\text{Im}\left[\frac{\lambda(z)}{\xi w(z)}\right] = b_j, \ j \in J = \{1, \ldots, 2K + 2n + 1\}, \tag{3.5}$$

where $a_j \in \Gamma_0 (j = 1, \ldots, 2K + 2n + 1)$ are distinct fixed points; and $b_j (j \in J)$ are all real constants satisfying the conditions

$$|b_j| \leq k_j, \ j \in J, \tag{3.6}$$

herein $k_j$ is a non-negative constant.

**Theorem 3.1.** Suppose that the first order complex equation (1.1) satisfies Condition C. Then any solution $w(z)$ of Problem $B_1$ for the complex equation (1.1) satisfies the estimates

$$C_0[\xi^2 w(z), \mathbb{D}] \leq M_1, \tag{3.7}$$

in which $\beta = \min(\alpha, 1 - 2/p_0), k = k(k_0, k_1, k_2, k_3), M_j = M_j(q_0, p_0, \beta, k, D), (j = 1, 2)$ are positive constants.

**Proof.** Similarly to the proof of Theorem 1.1, the solution $w(z)$ of Problem $B_1$ for (1.1) can be expressed in the formula as in (1.15), hence the boundary value problem $B_1$ can be transformed into the boundary value problem (Problem $B_1$) for analytic functions

$$\text{Re}\left[\frac{\lambda(z)}{\xi w(z)}\right] = r(z) + h(z), \ z \in \Gamma', \tag{3.8}$$

Where

$$h(z) = \begin{cases} 0, & \text{if } z \in \Gamma, \\ \beta, & \text{if } z \notin \Gamma, \end{cases} \tag{3.9}$$

And

$$\lambda(z) = \lambda[z(\zeta)]e^{\psi(z)}; \text{ } r(z) = r[z(\zeta)] - \text{Re}\left[\lambda[z(\zeta)]\Psi[z(\zeta)]e^{\psi(z)}\right],$$

$$a_j = \zeta(a_j), \ \beta_j - \text{Im}\left[\frac{\lambda(a_j)}{\xi w(a_j)}\right], \ j \in J$$
By (1.5), (1.9), it can be seen that $\Lambda(\zeta)$, $\xi(\zeta)$, $\hat{b}_j (j \in J)$ satisfy the conditions
\begin{equation}
C_{eq}[\Lambda(\zeta), L] \leq M_3, \ C_{eq}[\xi(\zeta), L] \leq M_3, \ |\hat{b}_j| \leq M_3, \ j \in J,
\end{equation}
Where $M_3 = M_3(q_0, p_0, \beta, k, D)$. If we can prove that the solution $\Phi(\zeta)$ of Problem $B_1$ satisfies the estimate
\begin{equation}
C_{eq}[\zeta^m\Phi(\zeta), \varpi] \leq M_4,
\end{equation}
in which $G = \zeta(D)$, $M_4 = M_4(q_0, p_0, \beta, k, D)$, then from the representation (3.3) of the solution $w(\zeta)$ and the estimates about $\Phi(\zeta)$, $\xi(\zeta)$ and its inverse function $z(\zeta)$, the estimates in (3.5) can be derived.

It remains to prove that (3.10) holds. For this, we first verify the boundedness of $\zeta^m\Phi(\zeta)$, i.e.
\begin{equation}
C[\zeta^m\Phi(\zeta), \varpi] \leq M_5 = M_5(q_0, p_0, \beta, k, D).
\end{equation}
Suppose that (3.11) is not true. Then there exist sequences of functions $\{\Lambda(\zeta)\}$, $\{\xi(\zeta)\}$, $\{\hat{b}_j\}$ satisfying the same conditions as $\Lambda(\zeta)$, $\xi(\zeta)$, $\hat{b}_j$, and $\Lambda(\zeta)$, $\xi(\zeta)$, $\hat{b}_j$ uniformly converge to $\Lambda(\zeta)$, $\xi(\zeta)$, $\hat{b}_j$ (j $\in J$) on L respectively. For the solution $\Phi(\zeta)$ of the boundary value problem (Problem $B_1$) corresponding to $\Lambda(\zeta)$, $\xi(\zeta)$, $\hat{b}_j$ (j $\in J$) we have $I_l = C[\Phi(\zeta), \varpi] \to \infty$ as $n \to \infty$. There is no harm in assuming that $I_l \geq 1, l = 1, 2, \ldots$. Obviously $\Phi(\zeta) = \Phi(\zeta)/I_l$ satisfies the boundary conditions
\begin{align*}
\text{Re}[\Lambda_l(\zeta) \Phi(\zeta)] &= |\hat{b}_j(\zeta) + h(\zeta)/I_l|, \zeta \in L^*, \\
\text{Im}[\Lambda_l(\zeta) \Phi(\zeta)] &= \hat{b}_j/I_l, \ j \in J,
\end{align*}
Applying the Schwarz formula, the Cauchy formula and the method of symmetric ex- tension (see Theorems 3.2-3.3, Chapter V, [3]), the estimate
\begin{equation}
C_{eq}[\zeta^m\Phi(\zeta), \varpi] \leq M_6
\end{equation}
can be obtained, where $M_6 = M_6(q_0, p_0, \beta, k, D)$. Thus we can select a subsequence of $\{\Phi_l(\zeta)\}$ which uniformly converge to an analytic function $\Phi_0(\zeta)$ in $G$, and $\Phi_0(\zeta)$ satisfies the homogeneous boundary conditions
\begin{align*}
\text{Re}[\Lambda_l(\zeta) \Phi_0(\zeta)] &= h(\zeta), \zeta \in L^*, \\
\text{Im}[\Lambda_l(\zeta) \Phi_0(\zeta)] &= 0, \ j \in J,
\end{align*}
On the basis of the uniqueness theorem, we conclude that $\Phi_0(\zeta) = 0, \zeta \in \varpi$. However, $C[\zeta^m\Phi_0(\zeta), \varpi] = 1$, from $C[\zeta^m\Phi_l(\zeta), \varpi] = 1$, it follows that there exists a point $\zeta \in \varpi$ such that $C[\zeta^m\Phi_0(\zeta)] = 1$. This contradiction proves that (3.11) holds. Afterwards using the method which leads from $C[\zeta^m\Phi_l(\zeta), \varpi] = 1$ to (3.12), the estimate (3.7) can be derived.

For verifying the existence of solutions of Problem B_1 for the complex equation (1.1), we need to add the following condition. For any continuous functions $w_1(z), w_2(z)$ in $D \setminus \{0\}$ and $[\zeta(z)]^T U(z) \in L^p(D)$, there is
\begin{equation}
F(z, w_1, U) - F(z, w_2, U) = \bar{Q}(z, w_1, w_2, U) U + \bar{A}(z, w_1, w_2, U)(w_1 - w_2),
\end{equation}
where $|\bar{Q}(z, w_1, w_2, U)| \leq q_0(\zeta(1)), L_p[|\bar{A}(z, w_1, w_2, U)|, D] \leq k_0$. When (1.1) is linear, (3.1) obviously holds. Moreover we first prove the existence of solutions of Problem B_1 for equation (1.1) with $F(z, w, w_c) = 0$ in $D_{1,m} = \{|z| < 1/m\} \cup \{|z| < a\} < 1/m\}$, i.e.
\begin{equation}
w_m = F_{1,m}(z, w, w_c), F_{1,m}(z, w, w_c) = \{w_{x}, w_{x}, w_{x}, w_{x}, \chi \in D_{m} = D \setminus D_{1/m}\}
\end{equation}
By the Leray-Schauder theorem, where $m$ is a sufficiently large positive integer.

**Theorem 3.2.** Suppose that equation (1.1) satisfies Condition C and (3.13). Then the singular Riemann-Hilbert problem (Problem B_1) for (3.14) has a solution.

**Proof.** In order to find a solution $w(z)$ of Problem B_1 for equation (3.14), we consider the equation (3.14) with the parameter $t \in [0, 1]$
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\[ w_2 = tF(z, w, w_z), \quad F(z, w, w_z) = Q_1 w_z + Q_3 w + A_1 w + A_2 w + A_3 \text{ in } D, \]

(3.15)

and introduce a bounded open set \( B_M \) of Banach space \( B = C_p(D_n) \cap \overline{L^1_{p0}}(D_n) \), whose elements are functions \( w(z) \) satisfying the condition

\[ w(z) \in C_p(D_n) \cap \overline{L^1_{p0}}(D_n) : C_p[w, D_n] + \overline{L^1_{p0}}[w, D_n] \]

(3.16)

where \( M_7 = 1 + M_1 + M_2 + M_3 \) are constants as similar to (3.7). We choose an arbitrary function \( W(z) \in \overline{B_M} \) and substitute it in the position of \( w \) in \( F(z, w, w_z) \). Applying the method in the proof of Theorem 1.1.2, [12], a solution \( w(z) = \Phi(z) + \Psi(z) = W(z) + tF(z) \) of Problem B1 for the complex equation

\[ w_{\overline{z}} = tF(z, W, W_z) \]

(3.17)

Can be found. Noting that \( tF(z, W, W_z) \in L_{p0}(\overline{D}) \), the above solution of Problem B1 for (3.17) is unique. Denoting by \( W(z) = \tilde{T}[W, t] (0 \leq t \leq 1) \) the mapping from \( W(z) \) to \( w(z) \), from Theorem 3.2, we know that if \( w(z) \) is a solution of Problem B for the equation

\[ w_{\overline{z}} = tF(z, w, w_z) \]

(3.18)

then the function \( w(z) \) satisfies the estimate

\[ C_p[w, D_n] < M_7 \]

(3.19)

Set \( B_0 = B_M \times [0, 1] \). In the following we verify the three conditions of the Leray-Schauder theorem:

(1) For every \( t \in [0, 1] \), \( \tilde{T}[W, t] \) continuously maps the Banach space \( B \) into itself, and is completely continuous in \( \overline{B_M} \). In fact, we arbitrarily select a sequence \( W_n(z) \) in \( \overline{B_M} \), \( n = 0, 1, 2, \ldots \), such that \( C_p[W_n - W_0, D_n] \rightarrow 0 \) as \( n \rightarrow \infty \). By Condition C, we see that \( L_{p0} [F(z, W_n, W_n)] \rightarrow 0 \) as \( n \rightarrow \infty \). Moreover, from \( W_n = \tilde{T}[W_n, t], \ w_0 = \tilde{T}[W_0, t] \), it is easy to see that \( w_n - w_0 \) is a solution of Problem B for the following complex equation

\[ (w_n - w_0)_{\overline{z}} = t[F(z, W_n, W_n) - F(z, W_0, W_0)] \text{ in } D, \]

(3.20)

and then we can obtain the estimate

\[ C_p[w_n - w_0, D_n] \leq 2k_0 C_p[w(z) - W_0(z), D_n]. \]

(3.21)

Hence \( C_p[W_n - W_0, D_n] \rightarrow 0 \) as \( n \rightarrow \infty \). In addition for \( W_n(z) \in \overline{B_M} \), \( n = 1, 2, \ldots \), we have \( w_n = \tilde{T}[W_n, t], w_0 = \tilde{T}[W_0, t] \), \( w_n, w_0 \in \overline{B_M} \), and then

\[ (w_n - w_0)_{\overline{z}} = t[F(z, W_n, W_n) - F(z, W_0, W_0)] \text{ in } D, \]

(3.22)

Where \( L_{p0} [F(z, W_n, W_n) - F(z, W_0, W_0)] \rightarrow 0 \) as \( n \rightarrow \infty \). Hence similarly to the proof of Theorem 3.1, we can obtain the estimate

\[ C_p[w_n - w_0, D_n] \leq M_8k_0 \]

(3.23)

where \( M_8 = M_0(q_{p0} p_0, \beta, k, D) \). Thus there exists a function \( w_0(z) \in \overline{B_M} \), from \( \{w_0(z)\} \) we can choose a subsequence \( \{w_{n_k}(z)\} \) such that \( C_p[w_{n_k} - W_0, D_n] \rightarrow 0 \) as \( k \rightarrow \infty \). This shows that \( w = \tilde{T}[W, t] \) is completely continuous in \( \overline{B_M} \). Similarly we can prove that for \( W(z) \in \overline{B_M} \), \( \tilde{T}[W, t] \) is uniformly continuous with respect to \( t \in [0, 1] \).

(2) For \( t = 0 \), it is evident that \( w = \tilde{T}[W, 0] = \Phi(z) \in B_M \).

(3) From the estimate (3.7), we see that \( w = \tilde{T}[W, t] (0 \leq t \leq 1) \) does not have a solution \( w(z) \) on the boundary \( \partial B_M = \overline{B_M} \setminus B_M \).

Hence by the Leray-Schauder theorem, we know that there exists a function \( w(z) \in \overline{B_M} \), such that \( w(z) = \tilde{T}[W(z), t] \), and the function \( w(z) \in C_p(D_n) \) is just a solution of Problem B for the complex equation (3.14).

**Theorem 3.3.** Suppose that equation (1.1) satisfies Condition C and (3.13). Then Problem B1 for (1.1) has a solution.

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Proof. According to Theorem 3.2, we have proved that Problem B₁ for (3.14) have a solution w₁(z), let m → ∞, we can derive that w₀(z) is the solution Problem B₁ for (1.1).

**Theorem 3.4.** Suppose that equation (1.1) satisfies Condition C and (3.13). Then Problem B_j (j=2, 3) for (1.1) have a unique solution.

**Proof.** We first verify the unique solvability of Problem B₃ for (1.1). As stated in the proof of Theorem 2.3, the boundary conditions (1.13) can be reduced to the following boundary conditions

\[ \text{Re} \{ \overline{\Lambda}(z) [(\zeta - \zeta(a))^{m} \zeta] W(z) \} = r(z) + h(z) \text{ in D}, \]

\[ \text{Im} \{ \overline{\Lambda}(a_j)(\zeta(a) - \zeta(a_j))^{m} W(a_j) \} = b_j, \quad j \in J, \]

where \( W(z) = w(z)/\Psi(z), \) \( \Psi(z) = (\zeta - \zeta(a))^{m}/\zeta \), \( b_j \) (\( j \in J \)) are real constants. It is easy to see \( W(z) \) satisfies the complex equation

\[ w_F = Q_1 w_z + Q_2 \overline{w}_F \Psi'(z) - A \overline{W} - [Q_2 \Psi'(z) - B(z)\Psi(z)]/\Psi \overline{W} + A_3 \Psi(z), \quad z \in D, \]

which index of \( \lambda(z)(\zeta(z) - \zeta(a))^{m}/\zeta^{n} \) on \( \Gamma \) equals to \( K + n - m(>0) \), by Theorem 3.3, the solvability of the boundary value problem (1.13) for (1.1) is verified.

Similarly we can prove the solvability of Problem B₂ for (1.1). From the solvability of Problem B₂ for (1.1), we can derive the existence of the homeomorphic solution for the nonlinear complex equation (1.1) with \( A(z, w) = B(z, w) = C(z, w) = 0 \) in \( D \) from the domain \( D \) mapping to the \( N+1 \)-connected rectilinear slit domain \( G \), the so-called \( N+1 \)-connected rectilinear slit domain means a domain whose boundary consists of \( N+1 \) rectilinear slits \( L_j (j = 0, 1, \ldots, N) \) with the oblique angles \( \theta_j \) (\( j = 0, 1, \ldots, N \)) respectively, where we must choose \( \lambda(z) = e^{i(\arg(b_j) + \pi/2)} \), \( \theta_j \) (\( j = 0, 1, \ldots, N \)) are real constants, in this case, the index \( K = 0 \).

Finally we give the conclusion in this paper, namely the singular Riemann-Hilbert problem with the nonnegative index for elliptic complex equations of first can be trans-formed into the non-singular Riemann-Hilbert problem with the nonnegative index for the corresponding complex equations of first order, due to we can handle the non-singular boundary value problem, then the corresponding results of non-singular boundary value problem can be derived.

**REFERENCES**


