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Gould Functional Recurrence Equations and Fibonacci Numbers

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Abstract: *The author has studied and solved different types of recurrence equations, especially non-linear. Some of his works in this area are mentioned in the references. In the present article a new type of nonlinear recurrence equations is introduced. These equations are named Gould functional recurrence equations and they have as solutions sequences of functions called Gould sequences. We prove that, in certain conditions, the Gould functional equations are equivalent to some numerical nonlinear recurrence equations whose solutions are the coefficients of the power series expansions of the solutions of the Gould equations. Then several particular Gould equations and their associated numerical recurrence equations will be solved. In all these cases, the closed formulas of their solutions are expressed by Fibonacci numbers. Some of the particular equations considered here have been already introduced by H. W. Gould, whence the origin of the name of the recurrence equations introduced in this article.*

Keywords: *Functional recurrence equations, Numerical recurrence relations, Gould sequences of functions, Fibonacci numbers.*
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1. INTRODUCTION

In the paper [8] H. W. Gould has considered some new nonlinear recurrence equations whose solutions are functions expressed relative to Fibonacci numbers. For one of these equations he has obtained a characterization related to the coefficients of power series development of the solutions.

The present paper aims at exploiting these ideas, considering a general nonlinear recurrence equation having functions as unknowns. This equation will extend the equations presented in the work [8] and therefore, it is called Gould functional recurrence equation. The sequence of its solutions is called Gould sequence of functions. Our general Gould equation is defined by an operator considered on the space of all functions, giving the connection between the unknown of the equation and the product of the two previous unknowns.

The notion defined here as Gould sequence of functions will receive a specific characterization which generalizes the one given in work [8]. This characterization consists in a numerical nonlinear recurrence relation satisfied by the coefficients of the power series developments of the functions by the Gould sequence.

The Gould functional equation and its associate numerical recurrence relation are solved in several particular cases. In all considered cases, some of them extending those from the work [8], the formulas of the solutions are expressed relative to Fibonacci numbers.

2. GOULD SEQUENCES OF FUNCTIONS

The notion of *Gould sequence of functions* is defines as

$$(u_n(x)) = G(A; u_0(x), u_1(x)),$$

given by an operator A on the space of all functions and the initial terms $u_0(x)$ and $u_1(x)$, as being the sequence that satisfies the nonlinear recurrence equation

$$u_{n+1}(x) = A(u_n(x)\ u_{n-1}(x)),\ n = 1,2\ldots. \tag{1}$$

called *Gould recurrence equation.* The word *functional* refers to the fact that the solution of such equation is a sequence of functions, in contrast with the recurrence relations whose solutions are numerical sequences and are therefore called *numerical recurrence relations* or *equations*.

For some A, $u_0(x)$ and $u_1(x)$, such sequences is considered in [8]. In this paper a characterization and examples of Gould sequences of functions are given by determining the formula of general term in each case. These formulas contain Fibonacci numbers,

$$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ \ldots,$$

defined by the well-known recurrence relation

$$F_{n+1} = F_n + F_{n-1},\quad n = 1,2\ldots. \tag{2}$$

3. The Characterization of the Gould Sequences

In this Section a linear continuous operator A is defined on a topological vector space E of functions. Let us suppose that $u_n(x) \in E$ and $x^n \in E$, with the following developments in convergent power series

$$u_n(x) = \sum_{k=0}^{\infty} a_k(n)x^k,\ A(x^n) = \sum_{k=0}^{\infty} t_k(n)x^k,\ n = 0,1,2\ldots. \tag{3}$$

Theorem. 3.1. *The sequence of functions* $u_n(x)$ *satisfies Gould recurrence equation* (1) *if and only if the coefficients* $a_k(n)$ *satisfy the recurrence relation*

$$a_k(n+1) = \sum_{j=0}^{\infty} t_k(j)\sum_{i=0}^{j} a_i(n)a_{j-i}(n-1),\ k = 0,1,2\ldots,\ n = 1,2\ldots, \tag{4}$$

The series of formula (4) being supposed to be convergent.

Proof. Taking into consideration the relations (3) and Cauchy's formula for series product, the recurrence equation (1) becomes

$$\sum_{k=0}^{\infty} a_k(n+1)x^k = A\left(\sum_{j=0}^{\infty} a_j(n)\,x^j\ \sum_{j=0}^{\infty} a_j(n-1)x^j\right) = A\left(\sum_{j=0}^{\infty}\sum_{i=0}^{j} a_i(n)a_{j-i}(n-1)x^j\right) =$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{j} a_i(n)a_{j-i}(n-1)A(x^j) = \sum_{j=0}^{\infty}\sum_{i=0}^{j} a_i(n)a_{j-i}(n-1)\sum_{k=0}^{\infty} t_k(j)x^k =$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} t_k(j)\sum_{i=0}^{j} a_i(n)a_{j-i}(n-1)x^k, \tag{5}$$

equivalent with the recurrence equation (4) .

Remark 1. *It should, however, be mentioned that the formula (4) occurs only if the two infinite series which appear in formula (5) from the proof of the theorem 3.1 can be interchangeable. Simple cases where this condition is fulfilled because one of these series is a finite sum, will be given in the following Section.*

4. Particular Cases of Gould Equations

1) If $A = I$, the identity operator, then $t_k(j) = \delta(j,k) = \begin{cases} 1 & , \quad j = k, \\ 0 & , \quad j \neq k, \end{cases}$ for $k, j = 0,1,2\ldots,$ the Kronecker symbol, hence the functional recurrence equation

$$u_{n+1}(x) = u_n(x)u_{n-1}(x) \tag{6}$$

and the numerical recurrence relation

$$a_k(n+1)=\sum_{i=0}^{k} a_i(n)a_{k-i}(n-1), \qquad (7)$$

for $k=0,1,2\ldots$, and $n=1,2\ldots$, are equivalent.

2) If $A=D$, the derivative operator in x, then

$$t_k(j)=(k+1)\delta(j,k+1)=\begin{cases} k+1 & , \quad j=k+1, \\ 0 & , \quad j\neq k+1, \end{cases}$$

hence from theorem 3.1 results that the functional recurrence equation

$$u_{n+1}(x)=D(u_n(x)u_{n-1}(x)) \qquad (8)$$

and the numerical recurrence relation

$$a_k(n+1)=(k+1)\sum_{i=0}^{k+1} a_i(n)a_{k+1-i}(n-1), \qquad (9)$$

for $k=0,1,2\ldots$, and $n=1,2,\ldots$,are equivalent.

3) If $A=x\,D$, then

$$t_k(j)=k\delta(j,k)=\begin{cases} k & , \quad j=k, \\ 0 & , \quad j\neq k, \end{cases}$$

Hence from theorem 3.1 results that the functional recurrence equation

$$u_{n+1}(x)=x\,D(u_n(x)u_{n-1}(x)) \qquad (10)$$

and the numerical recurrence relation

$$a_k(n+1)=k\sum_{i=0}^{k} a_i(n)a_{k-i}(n-1), \qquad (11)$$

for $k,n=1,2\ldots$, are equivalent.

Remark 2. *The above particular case* 3) *is given in* [8].

5. Examples of Gould Sequences of Functions

5.1 The Sequence $G(I;u_0(x),u_1(x))$

In this case, the recurrence equation (1) takes the form (6) and we have $u_2(x)=u_1(x)u_0(x)=u_1^{F_2}(x)u_0^{F_1}(x)$, $u_3(x)=u_2(x)u_1(x)=u_1^2(x)u_0(x)=u_1^{F_3}(x)u_0^{F_2}(x)$. For a fixed natural number n let us assume that

$$u_k(x)=u_1^{F_k}(x)u_0^{F_{k-1}}(x), \qquad (12)$$

for $1\leq k\leq n$. Then, using (6) and (2), we obtain

$$u_{n+1}(x)=u_1^{F_n}(x)u_0^{F_{n-1}}(x)u_1^{F_{n-1}}(x)u_0^{F_{n-2}}(x)=u_1^{F_n+F_{n-1}}(x)u_0^{F_{n-1}+F_{n-2}}(x)=u_1^{F_{n+1}}(x)u_0^{F_n}(x),$$

Hence formula (12) is true for $k=n+1$. In conformity with induction axiom, formula (12)is true for every natural number $k\neq 0$.

Remark 3. *If* $u_0(x)=u_1(x)$, *using again* (2), *it results that the solution of recurrence equation* (6) *is given by the formula*

$$u_k(x)=u_0^{F_{k+1}}(x),\ k=0,1,2,\ldots. \qquad (13)$$

Is called *discrete convolution* or *Cauchy product* of two numerical sequences (a_k) and (b_k), the sequence $(a_k)*(b_k)=\left(\sum_{i=0}^{k}a_ib_{k-i}\right)$, in entire papers the index k going through the set of natural numbers. Let us define the *convolution powers* of the sequence (a_k), by formula $(a_k)^{*n}=(a_k)*(a_k)*\cdots*(a_k)$, the product having n factors, for a natural number $n\geq 1$, and $(a_k)^{*0}=(1,0,0,\ldots)$, the convolution unit. With these notations, from (12), (13) and the Theorem 3.1, results the following

Theorem 5.1. *The numerical recurrence equation* (7), *which takes the form*

$$(a_k(n+1))=(a_k(n))*(a_k(n-1)),\ n=1,2,\ldots,\ k=0,1,2,\ldots, \qquad (14)$$

with the initial data $(a_k(0))$ *and* $(a_k(1))$, *has the solution given by the formula*

$$(a_k(n))=(a_k(1))^{*F_n}*(a_k(0))^{*F_{n-1}},\ n=1,2,\ldots,\ k=0,1,2,\ldots. \qquad (15)$$

If $a_k(0)=a_k(1),\ k=0,1,2,\ldots$, *then the solution of equation* (7) *is*

$$(a_k(n))=(a_k(0))^{*F_{n+1}},\ n=1,2,\ldots,\ k=0,1,2,\ldots. \qquad (16)$$

Remark 4. *We check now directly that the sequence* (15) *satisfy the equation* (14). *Indeed, using* (15) *and* (2), *we have*

$$(a_k(n))*(a_k(n-1))=(a_k(1))^{*F_n}*(a_k(0))^{*F_{n-1}}*(a_k(1))^{*F_{n-1}}*(a_k(0))^{F_{n-2}}=$$

$$=(a_k(1))^{*(F_n+F_{n-1})}*(a_k(0))^{*(F_{n-1}+F_{n-2})}=(a_k(1))^{*F_{n+1}}*(a_k(0))^{*F_n}=(a_k(n+1)),$$

hence the equation (14) *is satisfied. If* $a_k(0)=a_k(1)$, *using again* (2), *the solution* (15) *becomes*

$$(a_k(n))=(a_k(0))^{*F_n}*(a_k(0))^{*F_{n-1}}=(a_k(0))^{*(F_n+F_{n-1})}=(a_k(0))^{*F_{n+1}},$$

thus resulting formula (16).

5.2 The Sequence $G(D;1,e^x)$

In this case the recurrence equation (1) takes form (8)and

$$u_0(x)=1=e^{F_0x},\ u_1(x)=e^x=e^{F_1x},\ u_2(x)=D\left(e^{(F_0+F_1)x}\right)=D\left(e^{F_2x}\right)=F_2e^{F_2x}=e^{F_2x},$$

$$u_3(x)=D\left(e^{F_3x}\right)=F_3e^{F_3x},\ u_4(x)=F_3D\left(e^{F_4x}\right)=F_3F_4e^{F_4x},\ u_5(x)=F_3^2F_4D\left(e^{F_5x}\right)$$

$$=F_3^{F_3}F_4^{F_2}F_5^{F_1}e^{F_5x},\ u_6(x)=F_3^3F_4^2F_5D\left(e^{F_6x}\right)=F_3^{F_4}F_4^{F_3}F_5^{F_2}F_6^{F_1}e^{F_6x}.$$

For a fixed natural number n let us assume that

$$u_k(x)=F_3^{F_{k-2}}\cdots F_{k-2}^{F_3}F_{k-1}^{F_2}F_k^{F_1}e^{F_kx}, \qquad (17)$$

for $3\leq k\leq n$. Then, using (17), (8) and (2), we obtain

$$u_{n+1}(x)=D(u_n(x)u_{n-1}(x))=F_3^{F_{n-2}+F_{n-3}}\cdots F_{n-1}^{F_2+F_1}F_n^{F_1}D\left(e^{(F_n+F_{n-1})x}\right)=$$
$$=F_3^{F_{n-1}}\cdots F_{n-1}^{F_3}F_n^{F_2}D\left(e^{F_{n+1}x}\right)=F_3^{F_{n-1}}\cdots F_{n-1}^{F_3}F_n^{F_2}F_{n+1}^{F_1}e^{F_{n+1}x}.$$

In conformity with induction axiom, formula (17) is true for every natural number $k\geq 3$.

Remark 5. *The recurrence equation* (8) *with above initial values was considered in* [8] *but there was unfortunately wrong solved.*

From (17) and the Theorem 3.1, results in the following

Theorem 5.2. *The numerical recurrence relation* (9) *with initial data* $(a_k(0)) = (\delta(k,0))$ *and* $(a_k(1)) = \left(\frac{1}{k!}\right)$, *has the solution given by formulas*

$$(a_k(2)) = \left(\frac{1}{k!}\right),\ (a_k(3)) = \left(\frac{2^{k+1}}{k!}\right)$$

and

$$(a_k(n)) = \left(F_3^{F_{n-2}} \cdots F_{n-1}^{F_2} \frac{F_n^{k+1}}{k!}\right),\ n = 4,5,\ldots,\ k = 0,1,2,\ldots. \tag{18}$$

Remark 6. *Let us verify now that the solutions presented in the Theorem 5.2 satisfy the recurrence equation* (9). *Indeed, for* $n = 1,2,3,4$, *we have successively*

$$(k+1)\sum_{i=0}^{k+1} a_i(1)a_{k+1-i}(0) = (k+1)a_{k+1}(1) = \frac{1}{k!} = a_k(2),$$

$$(k+1)\sum_{i=0}^{k+1} a_i(2)a_{k+1-i}(1) = (k+1)\sum_{i=0}^{k+1}\frac{1}{i!(k+1-i)!} = \frac{1}{k!}\sum_{i=0}^{k+1}\binom{k+1}{i} = \frac{(1+1)^{k+1}}{k!} = \frac{2^{k+1}}{k!} = a_k(3),$$

$$(k+1)\sum_{i=0}^{k+1} a_i(3)a_{k+1-i}(2) = (k+1)\sum_{i=0}^{k+1}\frac{2^{i+1}}{i!}\frac{1}{(k+1-i)!} = \frac{2}{k!}\sum_{i=0}^{k+1}\binom{k+1}{i}2^i = 2\frac{(2+1)^{k+1}}{k!} =$$

$$= 2\frac{3^{k+1}}{k!} = F_3^{F_2}\frac{F_4^{k+1}}{k!} = a_k(4),$$

$$(k+1)\sum_{i=0}^{k+1} a_i(4)a_{k+1-i}(3) = (k+1)\sum_{i=0}^{k+1} 2\frac{3^{i+1}}{i!}\frac{2^{k+2-i}}{(k+1-i)!} = \frac{12}{k!}\sum_{i=0}^{k+1}\binom{k+1}{i}3^i 2^{k+1-i} =$$

$$= 12\frac{(3+2)^{k+1}}{k!} = 12\frac{5^{k+1}}{k!} = F_3^{F_3}F_4^{F_2}\frac{F_5^{k+1}}{k!} = a_k(5),$$

and for $n \geq 5$, *we have*

$$(k+1)\sum_{i=0}^{k+1} a_i(n)a_{k+1-i}(n-1) = (k+1)\sum_{i=0}^{k+1} F_3^{F_{n-2}}\cdots F_{n-1}^{F_2}\frac{F_n^{i+1}}{i!}F_3^{F_{n-3}}\cdots F_{n-2}^{F_2}\frac{F_{n-1}^{k+2-i}}{(k+1-i)!} =$$

$$= \frac{F_3^{F_{n-2}+F_{n-3}}\cdots F_{n-2}^{F_3+F_2}F_{n-1}^{F_2+1}F_n}{k!}\sum_{i=0}^{k+1}\binom{k+1}{i}F_n^i F_{n-1}^{k+1-i} =$$

$$= \frac{F_3^{F_{n-1}}\cdots F_{n-2}^{F_4}F_{n-1}^{F_3}F_n^{F_2}}{k!}(F_n + F_{n-1})^{k+1} = F_3^{F_{n-1}}\cdots F_{n-1}^{F_3}F_n^{F_2}\frac{F_{n+1}^{k+1}}{k!} = a_k(n+1).$$

5.3 The Sequence $G(D;1,x^m)$, With m Natural Number

Now let us also solve the recurrence equation (8), for $u_0(x) = 1 = x^{mF_0 - F_1 + 1}$ and $u_1(x) = x^m = x^{mF_1 - F_2 + 1}$. Using (8) and (2), we obtain

$$u_2(x) = D\left(x^{mF_1 - F_2 + 1}\right) = D\left(x^{mF_2 - F_3 + 2}\right) = (mF_2 - F_3 + 2)^{F_1} x^{mF_2 - F_3 + 1},$$

$$u_3(x) = (mF_2 - F_3 + 2)^{F_1} D\left(x^{m(F_1+F_2) - (F_2+F_3) + 2}\right) = (mF_2 - F_3 + 2)^{F_2} D\left(x^{mF_3 - F_4 + 2}\right) =$$

$$= (mF_2 - F_3 + 2)^{F_2}(mF_3 - F_4 + 2)^{F_1} x^{mF_3 - F_4 + 1}.$$

By mathematical induction it results

$$G(D;1,x^m)=\left(1\,,\,x^m\,,\prod_{k=2}^{n}(mF_k-F_{k+1}+2)^{F_{n+1-k}}x^{mF_n-F_{n+1}+1}\quad:\quad n=2,3,\ldots\right). \tag{19}$$

From (19) and the Theorem 3.1, results the following

Theorem 5.3. *The numerical recurrence equation (9) with the initial data* $(a_k(0))=(\delta(k,0))$ *and* $(a_k(1))=(\delta(k,m))$*, has the solution given by the formula*

$$(a_k(n))=\left(\prod_{j=2}^{n}(mF_j-F_{j+1}+2)^{F_{n+1-j}}\,\delta(k,mF_n-F_{n+1}+1)\right),\ n=2,3,\ldots. \tag{20}$$

Remark 7. *We will verify that the solution (20) satisfy the equation (9). Indeed,*

$$(k+1)\sum_{i=0}^{k+1}a_i(1)a_{k+1-i}(0)=(k+1)a_{k+1-m}(0)=$$
$$=(k+1)\delta(k+1-m,0)=m\delta(k,m-1)=a_k(2),$$
$$(k+1)\sum_{i=0}^{k+1}a_i(2)a_{k+1-i}(1)=(k+1)a_{k-m+2}(1)=$$
$$=(k+1)m\delta(k-m+2,m)=m(2m-1)\delta(k,2m-2)=a_k(3),$$
$$(k+1)\sum_{i=0}^{k+1}a_i(n)a_{k+1-i}(n-1)=(k+1)\prod_{j=2}^{n}(mF_j-F_{j+1}+2)^{F_{n+1-j}}a_{k-mF_n+F_{n+1}}(n-1)=$$
$$=(k+1)\prod_{j=2}^{n}(mF_j-F_{j+1}+2)^{F_{n+1-j}}\prod_{j=2}^{n-1}(mF_j-F_{j+1}+2)^{F_{n-j}}\,\delta(k-mF_n+F_{n+1},mF_{n-1}-F_n+1)=$$
$$=(k+1)(mF_n-F_{n+1}+2)^{F_1}\prod_{j=2}^{n-1}(mF_j-F_{j+1}+2)^{F_{n+1-j}+F_{n-j}}\,\delta(k,mF_{n+1}-F_{n+2}+1)=$$
$$=(mF_{n+1}-F_{n+2}+2)(mF_n-F_{n+1}+2)^{F_2}\prod_{j=2}^{n-1}(mF_j-F_{j+1}+2)^{F_{n+2-j}}\,\delta(k,mF_{n+1}-F_{n+2}+1)=$$
$$=\prod_{j=2}^{n+1}(mF_j-F_{j+1}+2)^{F_{n+2-j}}\,\delta(k,mF_{n+1}-F_{n+2}+1)=a_k(n+1).$$

5.4 The Sequence $G(xD;1,x^m)$

Let us solve recurrence equation (10) when $u_0(x)=1=x^{mF_0}$ and $u_1(x)=x^m=x^{mF_1}$. Using (10) and (2), we obtain

$$u_2(x)=xD(u_1(x)u_0(x))=xD(x^m)=mx^m=m^{F_3-1}F_2^{F_1}x^{mF_2},$$
$$u_3(x)=xD(u_2(x)u_1(x))=mxD(x^{2m})=2m^2x^{2m}=m^{F_4-1}F_2^{F_2}F_3^{F_1}x^{mF_3},$$
$$u_4(x)=xD(u_3(x)u_2(x))=m^{F_3+F_4-2}F_2^{F_1+F_2}F_3^{F_1}xD\left(x^{m(F_2+F_3)}\right)=$$
$$=m^{F_5-2}F_2^{F_3}F_3^{F_2}xD(x^{mF_4})=m^{F_5-1}F_2^{F_3}F_3^{F_2}F_4^{F_1}x^{mF_4}.$$

By mathematical induction it results in

$$G(xD;1,x^m)=\left(1\,,\,x^m\,,\,m^{F_{n+1}-1}\prod_{k=2}^{n}F_k^{F_{n+1-k}}x^{mF_n}\quad:\quad n=2,3,\ldots\right). \tag{21}$$

Remark 8. *For* $m=1$*, the recurrence equation* (10) *with the initial values* $u_0(x)=1$ *and* $u_1(x)=x$*, was considered in* [8]. From (21) and the Theorem 3.1, results the following

Theorem 5.4. *The numerical recurrence equation (11) with the initial data* $(a_k(0))=(\delta(k,0))$ *and* $(a_k(1))=(\delta(k,m))$*, has the solution given by the formula*

$$(a_k(n))=\left(1,1,m^{F_{n+1}-1}\prod_{j=2}^{n}F_j^{F_{n+1-j}}\delta(k,mF_n):n=2,3,\ldots\right). \tag{22}$$

Remark 9. *We will verify now that the solution (22) presented in the Theorem 5.4 satisfies the recurrence equation (11). Indeed, for* $n=1,2$*, we have successively*

$$k\sum_{i=0}^{k}a_i(1)a_{k-i}(0)=k\sum_{i=0}^{k}a_i(1)\delta(k-i,0)=ka_k(1)=k\delta(k,m)=m\delta(k,m)=a_k(2),$$

$$k\sum_{i=0}^{k}a_i(2)a_{k-i}(1)=k\sum_{i=0}^{k}a_i(2)\delta(k-i,m)=ka_{k-m}(2)km\delta(k-m,m)=2m^2\delta(k,2m)=a_k(3),$$

for $k\geq m$, the relation being also true, $0=0$, for $k<m$ and for $n\geq 2$,

$$k\sum_{i=0}^{k}a_i(n)a_{k-i}(n-1)=k\sum_{i=0}^{k}m^{F_{n+1}-1}\prod_{j=2}^{n}F_j^{F_{n+1-j}}\delta(i,mF_n)a_{k-i}(n-1)=$$

$$=km^{F_{n+1}-1}\prod_{j=2}^{n}F_j^{F_{n+1}-j}a_{k-mF_n}(n-1)=$$

$$=km^{F_{n+1}-1}\prod_{j=2}^{n}F_j^{F_{n+1-j}}m^{F_n-1}\prod_{j=2}^{n-1}F_j^{n-j}\delta(k-mF_n,mF_{n-1})=$$

$$=km^{F_{n+1}+F_n-2}F_n^{F_1}\prod_{j=2}^{n-1}F_j^{F_{n+1-j}+F_{n-j}}\delta(k,m(F_n+F_{n-1}))=$$

$$=km^{F_{n+2}-2}F_n^{F_1}\prod_{j=2}^{n-1}F_j^{F_{n+2-j}}\delta(k,mF_{n+1})=mF_{n+1}m^{F_{n+2}-2}F_n^{F_1}\prod_{j=2}^{n-1}F_j^{F_{n+2-j}}\delta(k,mF_{n+1})=$$

$$=m^{F_{n+2}-1}\prod_{j=2}^{n+1}F_j^{F_{n+2-j}}\delta(k,mF_{n+1})=a_k(n+1),$$

for $k\geq mF_n$*, the relation being true,* $0=0$*, if* $k<mF_n$*.*

5.5 The Sequence $G(xD;1,e^x)$.

Let us give form of the solution of the recurrence equation (10), when $u_0(x)=1$, and $u_1(x)=e^x=P_{F_2-1}(x)e^{F_1x}$, with $P_{F_2-1}(x)=P_0(x)=1$. In this section $P_n(x)$ denotes a polynomial of degree n.

Using (10) and (2), we obtain $u_2(x)=xD(e^x)=xe^x=P_{F_3-1}(x)e^{F_2x}$, with $P_{F_3-1}(x)=P_1(x)=x$ and $u_3(x)=xD(P_{F_3-1}(x)e^{F_3x})=xD(xe^{2x})=P_{F_4-1}(x)e^{F_3x}$, with $P_{F_4-1}(x)=P_2(x)=x(2x+1)$.For a fixed natural number n let us assume that

$$u_k(x)=P_{F_{k+1}-1}(x)e^{F_kx}, \tag{23}$$

for $k\leq n$. Then, using (23), (10) and (2), we obtain

$$u_{n+1}(x)=xD(u_n(x)u_{n-1}(x))=xD\left(P_{F_{n+1}-1}(x)e^{F_nx}P_{F_n-1}(x)e^{F_{n-1}x}\right)=$$
$$=xD\left(P_{F_{n+1}-1}(x)P_{F_n-1}(x)e^{(F_n+F_{n-1})x}\right)=xD\left(P_{F_{n+1}-1}(x)P_{F_n-1}(x)e^{F_{n+1}x}\right)=P_{F_{n+2}-1}(x)e^{F_{n+1}x}$$

where

$$P_{F_{n+2}-1}(x)=x\left[F_{n+1}P_{F_{n+1}-1}(x)P_{F_n-1}(x)+D\left(P_{F_{n+1}-1}(x)P_{F_n-1}(x)\right)\right],\ n=2,3,\ldots. \tag{24}$$

In conformity with induction axiom, formula (23) is true for every natural number k. The polynomials $P_{F_{n+1}-1}(x)$ can be calculated by recurrence from relation (24).

Example. For $n = 2,3,4$, in conformity with (24) we have

$$P_{F_4-1}(x) = P_2(x) = x\left[F_3 P_{F_3-1}(x) P_{F_2-1}(x) + D\left(P_{F_3-1}(x) P_{F_2-1}(x)\right)\right] = 2x^2 + x,$$

$$P_{F_5-1}(x) = P_4(x) = x\left[F_4 P_{F_4-1}(x) P_{F_3-1}(x) + D\left(P_{F_4-1}(x) P_{F_3-1}(x)\right)\right] = 6x^4 + 9x^3 + 2x^2,$$

$$P_{F_6-1}(x) = P_7(x) = x\left[F_5 P_{F_5-1}(x) P_{F_4-1}(x) + D\left(P_{F_5-1}(x) P_{F_4-1}(x)\right)\right] =$$

$$= 60x^7 + 192x^6 + 185x^5 + 62x^4 + 6x^3.$$

Remark 10. *The solution* (23) *of the functional recurrence equation* (10), *in case of the initial values* $u_0(x) = 1$ $u_1(x) = e^x$, *is mentioned without proof in* [8].

For $n = 0,1,2,\ldots$, let us consider the numerical sequence $(p_k(n))$, whose terms are the coefficients of the polynomial $P_{F_{n+1}-1}(x)$, when $0 \le k \le F_{n+1} - 1$ and are zeros, when $k > F_{n+1} - 1$. For example,

$(p_k(0)) = (p_k(1)) = (1,0,0,\ldots)$, $(p_k(2)) = (0,1,0,0,\ldots)$,
$(p_k(3)) = (0,1,2,0,0,\ldots)$, $(p_k(4)) = (0,0,2,9,6,0,0,\ldots)$,
$(p_k(5)) = (0,0,0,6,62,185,192,60,0,0,\ldots)$ and so on.

From (23), (24) and the Theorem 3.1, results the following

Theorem 5.5. *The numerical recurrence relation (11) with the initial data* $(a_k(0)) = (\delta(k,0))$ *and* $(a_k(1)) = \left(\dfrac{1}{k!}\right)$, *has the solution given by the formula*

$$(a_k(n)) = (p_k(n)) * \left(\frac{F_n^k}{k!}\right), \; n = 2,3,\ldots. \tag{25}$$

6. Discussions and Conclusions

Gould's article has opened a new path in the field of the recurrence equations and the present article is trying to revive this topic. It is very interesting that all the Gould equations have solutions which express themselves relative to the Fibonacci numbers. Finding a Gould sequence of functions that are not expressed relative to Fibonacci numbers remains an open problem.

By solving the Gould functional recurrence equations the solutions are also obtained for the associate numerical recurrence relations. In addition, we tested if the obtained solutions satisfy indeed these numerical recurrence relations. These tests are also interesting for algebraic work with Fibonacci's numbers.

Besides the classical functional recurrence equations, as those verified by the special functions, it is important to find new equations of this type as well as some methods of solving them. Such equations can be obtained through the combination of the classical equations. So, the Author has considered functional recurrence equations of algebraic-differential type in the works [1] and [3] and algebraic-integral in the work [5]. In order to solve such equations it could be useful the *hybrid Laplace transform*, introduced by V. Prepeliţă, [9], obtained through the combination between the usual Laplace transform and the Z-transform.

Another important issue is the application of these equations to the solution various problems. The Author has applied the recurrent equations to the combinatory theory in the papers [2] and [4] and to the probability in the paper [6].

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