International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume I, Issue I, August- 2013, PP 1-16
www.arcjournals.org

Margrabe Formulas for a Simple Bivariate Exponential Variance-Gamma Price Process (I) Theory

Werner Hürlimann

Wolters Kluwer Financial Services,
Seefeldstrasse 69,
CH-8008 Zürich, Switzerland.
werner.huerlimann@wolterskluwer.com

Received: 20-06-2013 **Revised: 21-07-2013** **Accepted: 24-07-2013**

Abstract: *For the purpose of simultaneous market consistent valuation of insurance liabilities and/or asset prices from financial markets, an alternative to the Black-Scholes-Vasicek deflator is derived. It is based on a simple multivariate exponential variance-gamma process associated to a multivariate Lévy process, whose components are drifted Brownian motions time changed by a common independent gamma subordinator. The logarithm of the multivariate variance-gamma deflator is linear in the variance-gamma components. The practicability of the state-price deflator approach is demonstrated by determining the price of the Margrabe exchange option with a bivariate exponential variance-gamma real-world or deflated price process. Applications to both Insurance and Finance are mentioned. In a specialized bivariate exponential variance-Erlang model, the formulas simplify to analytical closed-form expressions.*

Keywords: *Black-Scholes model, variance-gamma process, state-price deflator, market price of risk.*

1. INTRODUCTION

It has long been observed that insurance liability data and financial returns exhibit non-zero skewness, kurtosis and heavy tails, which cannot be captured by a Gaussian process. A typical example is the Black-Scholes-Merton lognormal model used in option pricing. Modern Actuarial Science and Finance is more and more focusing on alternative stochastic processes that enable the modeling of non-Gaussian characteristics. However, the construction of multivariate non-Gaussian processes for the simultaneous modeling of real-world insurance liabilities and/or asset prices from financial markets is a complex topic, which seldom leads to simple analytical formulas. Moreover, for market consistent valuation one needs to deflate these processes with so-called state price deflators or use an equivalent martingale measure for deflated price processes as argued in the Remarks 4.1. Though general frameworks for deriving state-price deflators exist (e.g. [1], [2]), there are not many papers, which propose explicit expressions for them and their corresponding distribution functions.

The goal of our contribution is two-fold. First, we propose an alternative to the multivariate Black-Scholes-Vasicek (BSV) deflator introduced in [3] (see also [4]). For the interested reader we remark that the article [5] contains an extension of the Black-Scholes deflator to a more general version with interest rates as additional source of randomness. From a mathematical viewpoint, it is natural to investigate other generalizations, namely the consideration of alternative asset price processes for use in incomplete financial markets. Indeed, let us assume that asset prices admit no arbitrage. Then, there exists a unique state-price deflator if, and only if, the market is complete. Otherwise, if the market is incomplete, several state-price deflators exist and pricing is not uniquely defined. Therefore, the study of state-price deflators is motivated by one of the main problems of Modern Finance, which consists to understand the pricing of arbitrary portfolios in incomplete markets.

A valuable and popular competitor to the ubiquitous multivariate exponential Gaussian process is the multivariate exponential variance-gamma process presented in Section 2. It is obtained from a multivariate Lévy process, whose components are drifted Brownian motions time changed by a common independent gamma subordinator. This process has been previously considered in [6] and [7]. For a better understanding of the state-price deflator approach, we recall briefly in Section 3 the construction of the BSV deflator. Proceeding similarly, the logarithm of the multivariate variance-gamma deflator is assumed to be linear in the variance-gamma components and its coefficients are explicitly determined in Theorem 4.2.

As second goal, the practicability of the present state-price deflator approach is demonstrated by pricing the Margrabe exchange option with a bivariate exponential variance-gamma real-world or deflated price process. Theorem 5.1 displays Margrabe formulas for the bivariate exponential variance-gamma real-world price process. Analytical closed-form expressions are obtained for the bivariate exponential variance-Erlang special case. Based on the VG deflator, we derive in Theorem 6.1 Margrabe formulas for the corresponding deflated price processes. Example 6.1 illustrates by comparing Margrabe option prices in the bivariate Black-Scholes model and the bivariate variance-Erlang model. Section 7 is devoted to further discussion and conclusions. Remaining technicalities are proved in Appendix 1 and 2. Statistical estimation of the multivariate variance-gamma model and an application to stock market indices are presented in [8].

2. A Simple Multivariate Variance-Gamma Process

In its original representation, the univariate VG process is defined as a drifted Brownian motion time changed by an independent gamma process. Viewed from the initial time 0 it is defined by

$$X_t = \theta \cdot G_t + \tau \cdot W_{G_t}, \quad t > 0, \tag{2.1}$$

where W_t is a standard Wiener process and the independent subordinator (i.e. an increasing, positive Lévy process) $G_t \sim \Gamma(\nu^{-1}t, \nu^{-1})$ is a gamma process with unit mean rate and variance rate ν. Since X_t is a Lévy process, the dynamics of the VG process is determined by its distribution at unit time. In fact, the random variable $X := X_{t=1} \sim VG(\theta, \tau, \nu)$ follows a three parameter distribution with cumulant generating function (cgf)

$$C_X(u) = \ln E[\exp(uX)] = -\nu^{-1} \cdot \ln(1 - \nu\theta u - \tfrac{1}{2}\nu\tau^2 u^2), \quad \tau, \nu > 0, -\infty < \theta < \infty. \tag{2.2}$$

One notes that the cgf is only defined over the open interval

$$-2(\sqrt{(\nu\theta)^2 + 2\nu\tau^2} - \nu\theta)^{-1} < u < 2(\sqrt{(\nu\theta)^2 + 2\nu\tau^2} + \nu\theta)^{-1}. \tag{2.3}$$

The formula (2.2) is obtained from the cgf of the gamma random variable $G := G_{t=1} \sim \Gamma(1/\nu, 1/\nu)$ by conditioning using that $X|G \sim N(\theta \cdot G, \tau^2 \cdot G)$ is normally distributed. Since the distribution of $VG(\theta, \tau, \nu)$ is infinitely divisible, the VG process has independent and stationary increments, which also follow a VG distribution, namely

$$X_{t+s} - X_s \sim VG(\theta t, \tau\sqrt{t}, \nu / t), \quad 0 \le s < t. \tag{2.4}$$

The symmetric case $\theta = 0$ is used in the original model by [9] and [10]. The VG process has been studied at many places (e.g. [11]-[14]). It is a special case of the bilateral gamma process and other tempered stable processes considered in many recent papers (e.g. [15]-[19]).

Several multivariate versions of the VG process have been considered so far. Madan and Seneta [9] first introduced a multivariate symmetric VG process by subordinating a multivariate Brownian motion without drift by a common gamma process. The asymmetric version of this model has been developed in Cont and Tankov [6] and Luciano and Schoutens [7]. Generalizing (2.1) these authors consider multivariate Lévy processes with VG components of the type

$$X_t^{(k)} = \theta_k \cdot G_t + \tau_k \cdot W_{G_t}^{(k)}, \quad k = 1,2,...,n, \tag{2.5}$$

where the $W_t^{(k)}$'s are *correlated* standard Wiener processes such that $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} dt$. More complex multivariate VG models have also been considered (e.g. [20]-[24]). Despite all its shortcomings, the use of the model (2.5) is justified theoretically by looking at the variance of its VG margins $X^{(k)} := X_{t=1}^{(k)} \sim VG(\theta_k, \tau_k, \nu)$ at unit time, namely

$$Var[X^{(k)}] = \tau_k^2 + \nu \cdot \theta_k^2. \tag{2.6}$$

Each variance decomposes into an *idiosyncratic* component τ_k^2, that is attributed to the Brownian motion, and an *exogenous* component $\nu \cdot \theta_k^2$, that is due to the gamma distributed time change of the Brownian motion. The parameters θ_k govern the exposures of the margins to the global *market uncertainty* measured by the common parameter ν. Similarly, one notes that the skewness and kurtosis are also affected by the single marginal settings and the common parameter ν. On the other hand, the statistical moment method developed in [8] and its successful application to real-world data justifies its use in practical work. To fix ideas, and for the sake of simplicity, the first focus is therefore on the model (2.5). The joint cgf of this multivariate process can be expressed in closed-form.

Proposition 2.1 (*cgf of the multivariate VG process*) The joint cgf of the multivariate VG process $X_t = (X_t^{(1)}, X_t^{(2)}..., X_t^{(n)}) \sim VG(\theta t, \Sigma t, \nu / t)$ with parameters $\theta = (\theta_1, \theta_2, ..., \theta_n), \Sigma = (\rho_{ij} \tau_i \tau_j)$, is determined by

$$C_{X_t}(u) = -\nu^{-1} t \cdot \ln\{1 - \nu \cdot (\theta^T u + \tfrac{1}{2} u^T \Sigma u)\}, \quad u = (u_1, u_2, ..., u_n). \tag{2.7}$$

Proof Since the conditional margins are normally distributed as $X_t^{(k)} | G_t \sim N(\theta_k G_t, \tau_k^2 G_t)$, one obtains the representation (2.7) from the following calculation

$$C_{X_t}(u) = \ln E[\exp(u^T X_t)] = \ln \mathrm{E}_{G_t}[\mathrm{E}[\exp(u^T X_t) | G_t]] = \ln \mathrm{E}_{G_t}[\exp(\theta^T u G_t + \tfrac{1}{2} u^T \Sigma u G_t)]. \ \Diamond$$

Remarks 2.1 The random vector at unit time $X = (X^{(1)}, X^{(2)}, ..., X^{(n)})$ of the special case $\nu = 1$ has a so-called multivariate asymmetric Laplace distribution, denoted $X \sim AL(\theta, \Sigma)$, that has been extensively studied in the monograph [26] (see also [27]). The parameter estimation of the shifted version $\xi + X \sim AL(\xi, \theta, \Sigma), \xi = (\xi_1, ..., \xi_n)$, has been discussed in [28] and [25]. The method is extended in [8] to the statistical estimation of the shifted variance-gamma model $\xi + VG(\theta, \Sigma, \nu)$.

3. THE BLACK-SCHOLES-VASICEK DEFLATOR

Recall the *Black-Scholes-Vasicek (BSV) deflator* introduced in [3]. Consider a multiple risk economy with $n \geq 1$ risky assets, whose real-world prices follow lognormal distributions. Given the current prices of these risky assets at initial time 0 we assume that the future prices at time $t > 0$ are described by exponential Brownian motions of the type

$$S_t^{(k)} = S_0^{(k)} \exp((m_t^{(k)} - \tfrac{1}{2} \sigma_k^2) t + v_t^{(k)} W_t^{(k)}), \quad k = 1,2,...,n, \tag{3.1}$$

where the $W_t^{(k)}$'s are correlated standard Wiener processes such that $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} dt$.

We assume that the correlation matrix $\Sigma = (\rho_{ij})$ is positive semi-definite with non-vanishing determinant. The quantities $m_t^{(k)}$ and $v_t^{(k)}$ are interpreted as *mean* and *standard deviation* per time unit of the k-th asset logarithmic return $r_t^{(k)}$ over the interval $[0,t]$, and σ_k is a constant volatility parameter. The representation (3.1) includes two popular asset pricing models:

Black-Scholes model:

$$dr_t^{(k)} = \mu_k dt + \sigma_k dW_t^{(k)} \quad \text{with} \quad m_t^{(k)} = \mu_k, \;\; v_t^{(k)} = \sigma_k.$$

Vasicek (Ornstein-Uhlenbeck) model:

$$dr_t^{(k)} = a_k(b_k - r_t^{(k)})dt + \sigma_k dW_t^{(k)} \quad \text{with} \quad m_t^{(k)} = \frac{b_k(1-e^{-a_k t})}{t}, \;\; v_t^{(k)} = \sigma_k\sqrt{\frac{1-e^{-2a_k t}}{2a_k t}}.$$

The economic model contains also a risk-free asset, which is assumed (for simplicity) to accumulate at a constant rate r per time unit. The *BSV deflator of dimension* n has the same form as the price processes in (3.1), i.e.

$$D_t = \exp(-\alpha_t t - \beta_t^T W_t), \quad t > 0, \tag{3.2}$$

for some time dependent parametric function α_t and vectors $\beta_t = (\beta_{t,1}, \beta_{t,2} ..., \beta_{t,n})^T$, $W_t = (W_t^{(1)}, W_t^{(2)} ..., W_t^{(n)})$. To define a state-price deflator the stochastic processes (3.1) and (3.2) must satisfy the martingale conditions

$$E[D_t] = e^{-rt}, \quad E[D_t S_t^{(k)}] = S_0^{(k)}, \quad t > 0, \quad k = 1,2,...,n, \tag{3.3}$$

where $E[\cdot]$ denotes conditional expectation with respect to the information at initial time 0.

Theorem 3.1 (*BSV deflator of dimension* n) Given is a financial market with a risk-free asset with constant rate of return r and $n \ge 1$ risky assets that have log-normal real-world prices (3.1). Assume a non-singular positive semi-definite correlation matrix $\Sigma = (\rho_{ij})$. Then, the BSV deflator (3.2) is determined by

$$D_t = \exp(-\alpha_t t - \beta_t^T W_t), \quad t > 0, \quad \text{with} \tag{3.4}$$

$$\begin{aligned} &\alpha_t = r + C_{W_{t=1}}(-\beta_t) = r + \tfrac{1}{2}\beta_t^T \Sigma \beta_t, \quad \beta_t = \Sigma^{-1}\lambda_t, \quad \lambda_t = (\lambda_{t,1}, \lambda_{t,2} ..., \lambda_{t,n})^T, \\ &\lambda_{t,k} v_t^{(k)} = m_t^{(k)} - r - \tfrac{1}{2}(\sigma_k^2 - [v_t^{(k)}]^2), \quad k = 1,2,...,n. \end{aligned} \tag{3.5}$$

The quantity $\lambda_{t,k}$ is called *market price of the* k*-th risky asset* at time t.

Proof The martingale conditions (3.3) are equivalent with the system of linear equations $-\alpha_t + r + \frac{1}{2}\beta_t^T \Sigma \beta_t = 0$ and $\Sigma\beta_t = \lambda_t$ (see [3], proof of Proposition 2). ◊

4. A State-Price Deflator for the Multivariate Exponential VG Process

We begin with the construction of the univariate VG deflator. Consider the following asset pricing model. Given the current price of a risky asset at time 0, its future price at time $t > 0$ is described by an exponential VG process

$$S_t = S_0 \exp((\mu - \omega)t + X_t), \quad X_t = \theta G_t + \tau W_{G_t}, \tag{4.1}$$

where μ represents the mean logarithmic rate of return of the risky asset per time unit and

$$\omega = C_X(1) = -\nu^{-1} \cdot \ln(1 - \nu(\theta + \tfrac{1}{2}\tau^2)) > 0. \tag{4.2}$$

To obtain (4.2) one uses the defining relationship $E[S_t] = S_0 \exp(\mu t)$ and the expression (2.7) for the univariate cgf. The *VG deflator* has the same form as the price process in (4.1). For some parameters α, β (to be determined) one sets

$$D_t = \exp(-\alpha t - \beta X_t), \quad t > 0. \tag{4.3}$$

Now, a simple cgf calculation shows that the state-price deflator martingale conditions

$$E[D_t] = e^{-rt}, \quad E[D_t S_t] = S_0, \quad t > 0, \tag{4.4}$$

are equivalent with the system of two non-linear equations in the three unknowns α, β, ω:

$$\begin{aligned} &-\alpha + r - \nu^{-1} \cdot \ln(1 + \nu\theta\beta - \tfrac{1}{2}\nu\tau^2\beta^2) = 0, \\ &-\alpha + \mu - \omega - \nu^{-1} \cdot \ln(1 - \nu\theta(1-\beta) - \tfrac{1}{2}\nu\tau^2(1-\beta)^2) = 0. \end{aligned} \tag{4.5}$$

Inserting the first equation into the second one yields the necessary relationship

$$\mu - r - \omega + \nu^{-1} \cdot \{\ln(1 + \nu\theta\beta - \tfrac{1}{2}\nu\tau^2\beta^2) - \ln(1 - \nu\theta(1-\beta) - \tfrac{1}{2}\nu\tau^2(1-\beta)^2)\} = 0. \tag{4.6}$$

As the system (4.5) has one degree of freedom, the unknown ω can be chosen arbitrarily, say

$$\omega = \mu - r, \tag{4.7}$$

which is interpreted as the (time-independent) *VG market price of the risky asset*. With the restriction (4.2) on the VG parameters this value is always positive. Inserted into (4.6) and using (4.2) shows that the parameter β is determined by the two alternate expressions

$$\beta\tau^2 = \theta + \tfrac{1}{2}\tau^2 = \nu^{-1}(1 - \exp(-\nu\omega)). \tag{4.8}$$

In particular, comparing the first and third term in (4.8) shows that the parameter β is a simple exponential transform of the market price, which is equivalent to the logarithmic relationship

$$\omega = \mu - r = -\nu^{-1} \cdot \ln(1 - \nu\beta\tau^2). \tag{4.9}$$

With the Mercator series for the logarithm, one sees that the VG market price of risk is given by

$$\omega = \mu - r = \nu^{-1} \cdot \sum_{j=1}^{\infty} \tfrac{1}{j}(\nu\beta\tau^2)^j = \beta\tau^2 + \tfrac{1}{2}\nu(\beta\tau^2)^2 + \ldots \approx \beta\tau^2 = \theta + \tfrac{1}{2}\tau^2, \tag{4.10}$$

where the last equality in the first order approximation follows from (4.8). Summarizing and rearranging the above one obtains the following VG deflator representation.

Theorem 4.1 (*univariate VG deflator*) Given is a risk-free asset with constant return r and a risky asset with real-world price (4.1). Then, the VG deflator (4.3) is determined by

$$D_t = \exp(-\alpha t - \beta(\beta - \tfrac{1}{2})\tau^2 \cdot G_t - \beta\tau \cdot W_{G_t}), \quad G_t \sim \Gamma(\nu^{-1}t, \nu^{-1}), \quad t > 0, \quad \text{with} \tag{4.11}$$

$$\alpha = r + C_X(-\beta) = r - \nu^{-1} \cdot \ln(1 - \tfrac{1}{2}\nu\beta(1-\beta)\tau^2), \quad \beta\tau^2 = \theta + \tfrac{1}{2}\tau^2. \tag{4.12}$$

Next, consider $n \geq 2$ risky assets. Given the current prices of these risky assets at initial time 0 their future prices at time $t > 0$ are described by exponential VG processes of the type

$$S_t^{(k)} = S_0^{(k)} \exp((\mu_k - \omega_k)t + X_t^{(k)}), \quad k = 1,2,\ldots,n, \tag{4.13}$$

where μ_k represents the mean logarithmic rate of return of the k-th risky asset per time unit, the random vector process $X_t = (X_t^{(1)},.X_t^{(2)},..,X_t^{(n)})$ follows a simple multivariate VG process with cgf (2.7), and similarly to (4.2) one has

$$\omega_k = -\nu^{-1} \cdot \ln(1 - \nu(\theta_k + \tfrac{1}{2}\tau_k^2)) > 0. \tag{4.14}$$

The *VG deflator of dimension* n has the same form as the price processes in (4.13). For some parameter α and vector $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ (both to be determined) one sets

$$D_t = \exp(-\alpha t - \beta^T X_t), \quad t > 0. \tag{4.15}$$

In this situation, the martingale conditions, which define the state-price deflator, are equivalent with the non-linear system of $n+1$ equations in the $2n+1$ unknowns $\alpha, \beta_k, \omega_k$ (use the explicit form of the cgf (2.7)):

$$\begin{aligned} &-\alpha + r - \nu^{-1} \cdot \ln\{1 + \nu \cdot (\theta^T \beta - \tfrac{1}{2} \beta^T \Sigma \beta)\} = 0, \\ &-\alpha + \mu_k - \omega_k - \nu^{-1} \cdot \ln\{1 - \nu \cdot (\theta^T \beta^{(k)} + \tfrac{1}{2} {\beta^{(k)}}^T \Sigma \beta^{(k)})\} = 0, \quad k = 1,2,...,n, \end{aligned} \tag{4.16}$$

with the vector notation $\beta^{(k)} = (\beta_1^{(k)}, \beta_2^{(k)}, ..., \beta_n^{(k)}), \beta_j^{(k)} = \delta_j^k - \beta_j, \; j,k = 1,2,...,n$. As the system (4.16) has n degrees of freedom, the unknowns ω_k can be chosen arbitrarily. A convenient appropriate choice, which leads to a simple solution of the system (4.16), consists to set

$$\omega_k = \mu_k - r, \tag{4.17}$$

which is interpreted as the (time-independent) *VG market price of the* k *-th risky asset*. Now, inserting the first equation of (4.16) into the second ones taking into account (4.17) yields

$$1 + \nu \cdot (\theta^T \beta - \tfrac{1}{2} \beta^T \Sigma \beta) = 1 - \nu \cdot (\theta^T \beta^{(k)} + \tfrac{1}{2} {\beta^{(k)}}^T \Sigma \beta^{(k)}), \quad k = 1,2,...,n. \tag{4.18}$$

A straightforward calculation shows that the components of the parameter vector β are determined by the two alternate expressions (use (4.14) for the second one)

$$\beta_k \tau_k^2 = \theta_k + \tfrac{1}{2} \tau_k^2 + \gamma_k \tau_k^2 = \nu^{-1}(1 - \exp(-\nu \omega_k)) + \gamma_k \tau_k^2, \gamma_k = \sum_{j \neq k} \rho_{kj} \frac{\tau_j}{\tau_k}, k = 1,2,...,n, \tag{4.19}$$

which generalize the relation (4.8). In particular, comparing the first and third term in (4.19) shows that the parameters β_k are simple exponential transforms of the market prices of risk, which are equivalent to the logarithmic relationships

$$\omega_k = -\nu^{-1} \cdot \ln(1 - \nu(\beta_k - \gamma_k)\tau_k^2), \quad k = 1,2,...,n. \tag{4.20}$$

Similarly to (4.10) one obtains the series expansions and the first order approximations for the VG market prices of the risky assets

$$\begin{aligned} \omega_k &= \nu^{-1} \cdot \sum_{j=1}^{\infty} \tfrac{1}{j} [\nu(\beta_k - \gamma_k)\tau_k^2]^j = (\beta_k - \gamma_k)\tau_k^2 + \tfrac{1}{2} \nu [(\beta_k - \gamma_k)\tau_k^2]^2 + ... \\ &\approx (\beta_k - \gamma_k)\tau_k^2 = \theta_k + \tfrac{1}{2} \tau_k^2, \end{aligned} \tag{4.21}$$

where the last equality follows from (4.19). Summarizing and rearranging the above one obtains the following multivariate VG deflator representation.

Theorem 4.2 (*VG deflator of dimension* n) Given are $n \geq 2$ risky assets with real-world prices (4.13), where the random vector process $X_t = (X_t^{(1)}, X_t^{(2)}, ..., X_t^{(n)})$ follows a multivariate VG process. Then, the VG deflator (4.15) is determined by

$$D_t = \exp(-\alpha t - \sum_{k=1}^{n} \beta_k X_t^{(k)}), \quad t > 0, \quad \text{with} \tag{4.22}$$

$$\alpha = r + C_{X_{t=1}}(-\beta), \quad \beta_k \tau_k^2 = \theta_k + (\tfrac{1}{2} + \gamma_k)\tau_k^2, \quad \gamma_k = \sum_{j \neq k} \rho_{kj} \frac{\tau_j}{\tau_k}, \quad k = 1,2,...,n. \tag{4.23}$$

Remarks 4.1 It is instructive to note that (3.2) and (4.15) correspond to an "Esscher transformed measure" that has long been used in option pricing (e.g. [29]). Moreover, an important connection with the standard no-arbitrage framework of Mathematical Finance must be mentioned (e.g. [30], Section 2.5, and [31], Chap. 2). By the Fundamental Theorem of Asset Pricing, the assumption of no-arbitrage (weak form of the efficient market hypothesis) is equivalent with the existence of an equivalent martingale measure for deflated price processes. In complete markets, the equivalent martingale measure is unique, perfect replication of contingent claims holds, and straightforward pricing applies. In incomplete markets, an economic model is required to decide upon which equivalent martingale measure is appropriate. Now, let P denotes the real-world measure and P^* an equivalent martingale measure. Then, one can either work under P, where the price processes are deflated with a state-price deflator. Alternatively, one can work under P^* by discounting the price processes with the bank account numeraire. Working with financial instruments only, one often works under P^*. But, if additionally insurance liabilities are considered, one works under P (see [30], Remark 2.13). A non-trivial example is pricing of the "guaranteed maximum inflation death benefit (GMIDB) option" (equation (5.4) in [4]). In the present paper, we demonstrate the practicability of the state-price deflator approach for exponential variance-gamma price processes of Margrabe type used in insurance and financial markets. In insurance, the Margrabe option has been discussed by [32], [33], [34], Sections V.4 to V.6, [30], Section 4. On the other hand, [8] offers an alternative approach to calibration based on a novel multivariate statistical moment method that uses besides means and covariances also the coskewness and cokurtosis tensors. A real-world comparison of the exponential variance-gamma model with the ubiquitous Black-Scholes model for pricing the Margrabe option is also made.

5. MARGRABE FORMULA FOR THE REAL-WORLD PRICE PROCESS

Given is the shifted version $\xi \cdot t + X_t$, $\xi = (\xi_1, \xi_2, ..., \xi_n)$, of the Lévy process induced by the unit time shifted multivariate VG random vector $\xi + X \sim VG(\xi, \theta, \Sigma, \nu)$. We assume that future values of financial entities at time $t > 0$ are described by exponential VG processes of the form

$$S_t^{(k)} = \exp(\xi_k t + X_t^{(k)}), \quad k = 1,.2,..,n. \tag{5.1}$$

This general framework enables simultaneous modelling of real-world insurance liabilities and/or asset prices from financial markets. For market consistent valuation one needs to deflate them with so-called state price deflators, as explained in the Remarks 4.1, which will be done in the next Sections. In both settings we derive integral and closed-form formulas for the two-dimensional Margrabe exchange option, here of the real-world type

$$M_t^R = E[(S_t^{(1)} - S_t^{(2)})_+], \quad t > 0. \tag{5.2}$$

Therefore, the bivariate case $n = 2$ in (5.1) is of interest and for this we set $\rho = \rho_{12}$. For simplicity and without loss of generality, it suffices to discuss the unit time case $t = 1$. By abuse of notation, the time index is removed from any stochastic process in the following. Conditioning on the common gamma subordinator, one can write

$$\begin{aligned} M^R &= E[(S^{(1)} - S^{(2)})_+] = \int_0^\infty M^R(w) f(w) dw, \\ M^R(w) &= E[(S^{(1)} - S^{(2)})_+ | G = w], \quad f(w) = \gamma \cdot (\gamma w)^{\gamma-1} e^{-\gamma w} / \Gamma(\gamma), \quad \gamma = \nu^{-1}. \end{aligned} \tag{5.3}$$

As usual $\Phi(x)$ denotes the standard normal distribution, $\overline{\Phi}(x) = 1 - \Phi(x)$ is the survival function, and $\varphi(x) = \Phi'(x)$ is the density. The bivariate standard normal density is denoted by

$$\varphi_2(x, y; \rho) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right\}.$$

By definition (2.5) one obtains

$$M^R(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{\xi_1 + \theta_1 w + \tau_1 \sqrt{w} x} - e^{\xi_2 + \theta_2 w + \tau_2 \sqrt{w} y})_+ \varphi_2(x, y; \rho) dx dy. \tag{5.4}$$

The expression in the bracket of (5.4) is non-negative provided $x \ge x(y)$ with $x(y) = \frac{\xi_2 - \xi_1}{\tau_1 \sqrt{w}} + \frac{(\theta_2 - \theta_1)}{\tau_1}\sqrt{w} + \frac{\tau_2}{\tau_1} y$. Now, it is possible to separate the double integral as $M^R(w) = \int_{-\infty}^{\infty} J^R(y,w)\varphi(y)dy$, because $\varphi_2(x,y;\rho) = \varphi(y)\varphi((x-\rho y)/\sqrt{1-\rho^2})/\sqrt{1-\rho^2}$. With this the inner integral reads

$$J^R(y,w) = (1/\sqrt{1-\rho^2}) \cdot \int_{x(y)}^{\infty} \{e^{\xi_1 + \theta_1 w + \tau_1 \sqrt{w}x} - e^{\xi_2 + \theta_2 w + \tau_2 \sqrt{w}y}\}\varphi((x-\rho y)/\sqrt{1-\rho^2})dx. \tag{5.5}$$

A straightforward application of Lemma A1.1 in the Appendix 1 yields

$$J^R(y,w) = e^{\xi_1 + \theta_1 w + \rho\tau_1\sqrt{w}y + \frac{1}{2}(1-\rho^2)\tau_1^2 w}\Phi\left(\tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}\,\tau_1\sqrt{w}\right) - e^{\xi_2 + \theta_2 w + \tau_2\sqrt{w}y}\Phi\left(\tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}}\right).$$

Taking into account the above definition of the auxiliary function $x(y)$, one rewrites the arguments inside the normal distribution functions as

$$\tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}\,\tau_1\sqrt{w} = a + cy, \quad \tfrac{\rho y - x(y)}{\sqrt{1-\rho^2}} = b + cy, \text{ with}$$

$$a = \tfrac{(\theta_1 - \theta_2 + (1-\rho^2)\tau_1^2)w + \xi_1 - \xi_2}{\tau_1\sqrt{w}\sqrt{1-\rho^2}}, \quad b = \tfrac{(\theta_1-\theta_2)w + \xi_1 - \xi_2}{\tau_1\sqrt{w}\sqrt{1-\rho^2}}, \quad c = \tfrac{\rho\tau_1 - \tau_2}{\tau_1\sqrt{1-\rho^2}}.$$

Furthermore, one notes that

$$e^{\rho\tau_1\sqrt{w}y}\varphi(y) = e^{\frac{1}{2}\rho^2\tau_1^2 w}\varphi(y - \rho\tau_1\sqrt{w}), \quad e^{\tau_2\sqrt{w}y}\varphi(y) = e^{\frac{1}{2}\tau_2^2 w}\varphi(y - \tau_2\sqrt{w}).$$

Now, using twice the Lemma A1.2 of the Appendix 1 one obtains

$$M^R(w) = e^{\xi_1 + (\theta_1 + \frac{1}{2}\tau_1^2)w}\Phi\left(\tfrac{a + c\rho\tau_1\sqrt{w}}{\sqrt{1+c^2}}\right) - e^{\xi_2 + (\theta_2 + \frac{1}{2}\tau_2^2)w}\Phi\left(\tfrac{b + c\tau_2\sqrt{w}}{\sqrt{1+c^2}}\right).$$

An algebraic calculation based on the expressions for the coefficients a, b, c yields further

$$\begin{gathered} M^R(w) = e^{\xi_1 + (\theta_1 + \frac{1}{2}\tau_1^2)w}\Phi\left(\tfrac{\theta_1 - \theta_2 + \omega_1^2}{\omega}\sqrt{w} + \tfrac{\xi_1 - \xi_2}{\omega\sqrt{w}}\right) - e^{\xi_2 + (\theta_2 + \frac{1}{2}\tau_2^2)w}\Phi\left(\tfrac{\theta_1 - \theta_2 - \omega_2^2}{\omega}\sqrt{w} + \tfrac{\xi_1 - \xi_2}{\omega\sqrt{w}}\right), \\ \omega_k^2 = \tau_k^2 - \rho\tau_1\tau_2, \quad k = 1,2, \quad \omega = \sqrt{\omega_1^2 + \omega_2^2}. \end{gathered} \tag{5.6}$$

We are ready for the following result.

Theorem 5.1 (*Margrabe formula for the real-world exponential shifted bivariate VG process*) Given is the bivariate process at unit time $S^{(k)} = \exp(\xi_k + X^{(k)}), k = 1,2$, with $\xi + X \sim VG(\xi, \theta, \Sigma, \nu)$. With $\rho = \rho_{12}, \gamma = \nu^{-1}$, $\omega_k^2 = \tau_k^2 - \rho\tau_1\tau_2, k = 1,2, \omega = \sqrt{\omega_1^2 + \omega_2^2}$, one has

$$\begin{gathered} M^R = E[(S^{(1)} - S^{(2)})_+] = e^{\xi_1} \cdot \Psi(a_1, b_1, c, \gamma) - e^{\xi_2} \cdot \Psi(a_2, b_2, c, \gamma), \\ \Psi(a,b,c,\gamma) = 1/\Gamma(\gamma) \cdot \int_0^{\infty} z^{\gamma-1}e^{-z}e^{az}\Phi(b\sqrt{z} + c/\sqrt{z})dz, \\ a_k = \gamma^{-1}(\theta_k + \tfrac{1}{2}\tau_k^2), \quad b_k = \frac{\theta_1 - \theta_2 + (-1)^{k-1}\omega_k^2}{\omega\sqrt{\gamma}}, \quad k = 1,2, \quad c = \frac{\sqrt{\gamma}(\xi_1 - \xi_2)}{\omega}. \end{gathered} \tag{5.7}$$

Proof It remains to insert (5.6) into (5.3) and make the change of variables $z = \gamma \cdot w$. ◊

Remark 5.1 One observes that the Ψ-function in (5.7) is related to the one stated in [26], p.296 (European risk-neutral call-option price for an exponential VG price process).

A particular instance, for which the formula (5.7) simplifies considerably, is the special case $c = 0$, that is $\xi_1 = \xi_2$, which is analyzed into more details. In this situation, the Ψ-function simplifies to the calculation of the expression (make the change of variables $z = x^2 / 2(1-a)$ under the assumption $a < 1$)

$$\Psi(a,b,0,\gamma) = \frac{2\cdot[2(1-a)]^{-\gamma}}{\Gamma(\gamma)} \cdot L_\gamma\left(\frac{b}{\sqrt{2(1-a)}}\right), \tag{5.8}$$

with the integral function of normal type

$$L_\gamma(z) = \sqrt{2\pi} \cdot \int_0^\infty x^{2\gamma-1}\varphi(x)\Phi(z\cdot x)dx. \tag{5.9}$$

The assumption $a < 1$ in (5.8) means that (5.7) will be finite for $c = 0$ provided $a_k = \gamma^{-1}(\theta_k + \frac{1}{2}\tau_k^2) < 1,\ k = 1,2$, a condition, which is equivalent with the existence of the cgf $C_{SG}(a_k) = -\gamma\cdot\ln(1-a_k)$ of a standard gamma random variable with shape parameter γ and scale parameter 1. In case the gamma distributed subordinator $G_t \sim \Gamma(\gamma\cdot t,\gamma)$ reduces to an Erlang distributed subordinator $G_t \sim Erlang(m\cdot t, m)$ with integer parameter $m = 1,2,\ldots$, the integral function (5.9) can be evaluated according to the following closed-form formula (proof in Appendix 2):

$$L_m(d) = 2^{m-2}(m-1)!\left(1 + \tfrac{d}{\sqrt{1+d^2}}\left(1 + \sum_{k=1}^{m-1}\frac{(2k-1)!!}{2^k k!(1+d^2)^k}\right)\right), \quad m = 1,2,\ldots. \tag{5.10}$$

6. Margrabe Formula for the Deflated Price Process

Consider the Margrabe exchange option for the deflated price process in (5.1) such that

$$M_t^D = E[D_t(S_t^{(1)} - S_t^{(2)})_+], \quad t > 0. \tag{6.1}$$

The state-price deflator is of the form (4.22) with $n = 2$. As in Section 5, we set $\rho = \rho_{12}$ and discuss the unit time case $t = 1$. Conditioning on the common gamma subordinator, one writes

$$\begin{aligned} &M^D = E[D(S^{(1)} - S^{(2)})_+] = \int_0^\infty M^D(w)f(w)dw, \\ &M^D(w) = E[D(S^{(1)} - S^{(2)})_+ | G = w], \quad f(w) = \gamma\cdot(\gamma w)^{\gamma-1}e^{-\gamma w}/\Gamma(\gamma), \quad \gamma = \nu^{-1}. \end{aligned} \tag{6.2}$$

Similarly to (5.5) it is possible to separate the double integral as $M^D(w) = \int_{-\infty}^\infty J^D(y,w)\varphi(y)dy$ with the inner integral

$$\begin{aligned} &J^D(y,w) = 1/\sqrt{1-\rho^2}\int_{x(y)}^\infty \left\{\begin{matrix} e^{\xi_1-\alpha+(1-\beta_1)(\theta_1 w+\tau_1\sqrt{w}x)-\beta_2(\theta_2 w+\tau_2\sqrt{w}y)} \\ -e^{\xi_2-\alpha-\beta_1(\theta_1 w+\tau_1\sqrt{w}x)+(1-\beta_2)(\theta_2 w+\tau_2\sqrt{w}y)} \end{matrix}\right\}\varphi((x-\rho y)/\sqrt{1-\rho^2})dx, \\ &x(y) = \frac{\xi_2-\xi_1}{\tau_1\sqrt{w}} + \frac{(\theta_2-\theta_1)}{\tau_1}\sqrt{w} + \frac{\tau_2}{\tau_1}y. \end{aligned} \tag{6.3}$$

A straightforward application of Lemma A1.1 in the Appendix 1 yields

$$\begin{aligned} J^D(y,w) = {} & e^{\xi_1-\alpha+(1-\beta_1)\theta_1 w-\beta_2(\theta_2 w+\tau_2\sqrt{w}y)+(1-\beta_1)\rho\tau_1\sqrt{w}y+\frac{1}{2}(1-\rho^2)(1-\beta_1)^2\tau_1^2 w}\Phi\left(\tfrac{\rho y-x(y)}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}(1-\beta_1)\tau_1\sqrt{w}\right) \\ & - e^{\xi_2-\alpha-\beta_1\theta_1 w+(1-\beta_2)(\theta_2 w+\tau_2\sqrt{w}y)-\beta_1\rho\tau_1\sqrt{w}y+\frac{1}{2}(1-\rho^2)\beta_1^2\tau_1^2 w}\Phi\left(\tfrac{\rho y-x(y)}{\sqrt{1-\rho^2}} - \sqrt{1-\rho^2}\beta_1\tau_1\sqrt{w}\right). \end{aligned}$$

Taking into account the definition of the auxiliary function $x(y)$, one rewrites the arguments inside the normal distribution functions as

$\frac{\rho y - x(y)}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}(1-\beta_1)\tau_1\sqrt{w} = a + cy, \quad \frac{\rho y - x(y)}{\sqrt{1-\rho^2}} - \sqrt{1-\rho^2}\beta_1\tau_1\sqrt{w} = b + cy,$ with

$a = \frac{(\theta_1 - \theta_2 + (1-\rho^2)(1-\beta_1)\tau_1^2)w + \xi_1 - \xi_2}{\tau_1\sqrt{w}\sqrt{1-\rho^2}}, \quad b = \frac{(\theta_1 - \theta_2 - (1-\rho^2)\beta_1\tau_1^2)w + \xi_1 - \xi_2}{\tau_1\sqrt{w}\sqrt{1-\rho^2}}, \quad c = \frac{\rho\tau_1 - \tau_2}{\tau_1\sqrt{1-\rho^2}}.$

Furthermore, one notes that

$$e^{\{(1-\beta_1)\rho\tau_1 - \beta_2\tau_2\}\sqrt{w}y}\varphi(y) = e^{\frac{1}{2}\{(1-\beta_1)\rho\tau_1 - \beta_2\tau_2\}^2 w}\varphi(y - \{(1-\beta_1)\rho\tau_1 - \beta_2\tau_2\}\sqrt{w}),$$
$$e^{\{(1-\beta_2)\tau_2 - \beta_1\rho\tau_1\}\sqrt{w}y}\varphi(y) = e^{\frac{1}{2}\{(1-\beta_2)\tau_2 - \beta_1\rho\tau_1\}^2 w}\varphi(y - \{(1-\beta_2)\tau_2 - \beta_1\rho\tau_1\}\sqrt{w}).$$

Now, using twice the Lemma A1.2 of the Appendix 1 one obtains

$$\begin{aligned} M^D(w) &= e^{\xi_1 - \alpha + \left\{(1-\beta_1)\tau_1 + \frac{1}{2}(1-\beta_1)^2\tau_1^2 - \beta_2(\theta_2 + (1-\beta_1)\rho\tau_1\tau_2) + \frac{1}{2}\beta_2^2\tau_2^2\right\}w}\Phi\left(\tfrac{a + c\{(1-\beta_1)\rho\tau_1 - \beta_2\tau_2\}\sqrt{w}}{\sqrt{1+c^2}}\right) \\ &- e^{\xi_2 - \alpha + \left\{(1-\beta_2)\tau_2 + \frac{1}{2}(1-\beta_2)^2\tau_2^2 - \beta_1(\theta_1 + (1-\beta_2)\rho\tau_1\tau_2) + \frac{1}{2}\beta_1^2\tau_1^2\right\}w}\Phi\left(\tfrac{b + c\{(1-\beta_2)\tau_2 - \beta_1\rho\tau_1\}\sqrt{w}}{\sqrt{1+c^2}}\right). \end{aligned} \tag{6.4}$$

Repeated use of the following relationships for β_1, β_2 in (3.23) is made:

$$\beta_k\tau_k^2 = \theta_k + \tfrac{1}{2}\tau_k^2 + \rho\tau_1\tau_2, \quad k = 1,2. \tag{6.5}$$

An algebraic calculation based on the expressions for the coefficients a,b,c yields

$$\begin{aligned} &\tfrac{a + c\{(1-\beta_1)\rho\tau_1 - \beta_2\tau_2\}\sqrt{w}}{\sqrt{1+c^2}} = b_1\sqrt{w} + \tfrac{\xi_1 - \xi_2}{\omega\sqrt{w}}, \quad b_1 = \tfrac{1}{2}\omega + (\beta_1 - \beta_2)\tfrac{\rho\tau_1\tau_2}{\omega}, \\ &\tfrac{b + c\{(1-\beta_2)\tau_2 - \beta_1\rho\tau_1\}\sqrt{w}}{\sqrt{1+c^2}} = b_2\sqrt{w} + \tfrac{\xi_1 - \xi_2}{\omega\sqrt{w}}, \quad b_2 = -\tfrac{1}{2}\omega + (\beta_1 - \beta_2)\tfrac{\rho\tau_1\tau_2}{\omega}, \\ &\omega_k^2 = \tau_k^2 - \rho\tau_1\tau_2, \quad k = 1,2, \quad \omega = \sqrt{\omega_1^2 + \omega_2^2}. \end{aligned} \tag{6.6}$$

Furthermore, the coefficients of w in the curly brackets of (6.4) are respectively equal to

$$a_1 = (1-\beta_2)\rho\tau_1\tau_2 - \theta^T\beta + \tfrac{1}{2}\beta^T\Sigma\beta, \quad a_2 = (\beta_1 - 1)\rho\tau_1\tau_2 - \theta^T\beta + \tfrac{1}{2}\beta^T\Sigma\beta. \tag{6.7}$$

We are ready for the following result.

Theorem 6.1 (*Margrabe formula for the deflated exponential shifted bivariate VG process*) Given is the bivariate process at unit time $S^{(k)} = \exp(\xi_k + X^{(k)}), k = 1,2$, with $\xi + X \sim VG(\xi, \theta, \Sigma, \nu)$, the bivariate deflator $D = \exp(-\alpha - \beta_1 X^{(1)} - \beta_2 X^{(2)})$ of Theorem 4.2, and set $\rho = \rho_{12}, \gamma = \nu^{-1}$, $\omega_k^2 = \tau_k^2 - \rho\tau_1\tau_2, k = 1,2, \omega = \sqrt{\omega_1^2 + \omega_2^2}$. Then one has

$$\begin{aligned} &M^D = E[D(S^{(1)} - S^{(2)})_+] = e^{\xi_1 - \alpha}\cdot\Psi(a_1, b_1, c, \gamma) - e^{\xi_2 - \alpha}\cdot\Psi(a_2, b_2, c, \gamma), \\ &\Psi(a,b,c,\gamma) = 1/\Gamma(\gamma)\cdot\int_0^\infty z^{\gamma-1}e^{-z}e^{az}\Phi(b\sqrt{z} + c/\sqrt{z})dz, \\ &a_1 = \gamma^{-1}((1-\beta_2)\rho\tau_1\tau_2 - \theta^T\beta + \tfrac{1}{2}\beta^T\Sigma\beta), \quad a_2 = \gamma^{-1}((\beta_1 - 1)\rho\tau_1\tau_2 - \theta^T\beta + \tfrac{1}{2}\beta^T\Sigma\beta), \\ &b_k = \sqrt{\gamma^{-1}}\cdot(\tfrac{1}{2}(-1)^{k-1}\omega + (\beta_1 - \beta_2)\frac{\rho\tau_1\tau_2}{\omega}), \quad k = 1,2, \quad c = \frac{\sqrt{\gamma}(\xi_1 - \xi_2)}{\omega}. \end{aligned} \tag{6.8}$$

Proof It remains to insert (6.4) into (6.2), make the necessary identifications based on (6.6) and (6.7), and perform the change of variables $z = \gamma\cdot w$. ◊

In the special case $c = 0$, that is $\xi_1 = \xi_2$, and if $a_k < 1, k = 1,2$, the expression (6.8) can be evaluated using the formula (5.8) for the Ψ-function. Specializing further to a shifted simple bivariate variance-Erlang price process with integer parameter $\gamma = m = 1,2,\ldots$, one obtains via (5.10) closed-form Margrabe formulas.

Example 6.1 Bivariate Black-Scholes versus bivariate variance-Erlang models

It is interesting to compare the classical closed-form formula by Margrabe [35] with closed-form variants obtained from (6.8). In the bivariate Black-Scholes model the future prices at unit time $t=1$ are described by the equations (3.1), that is

$$S_1^{(k)} = S_0^{(k)} \exp(\mu_k - \tfrac{1}{2}\sigma_k^2 + \sigma_k Z_k), \quad k = 1,2, \tag{6.9}$$

where the Z_k's are correlated standard normal with correlation coefficient ρ^{BS}. The means and variance-covariance characteristics of (6.9) are given by

$$\begin{aligned} E_i^{BS} &= E[S_1^{(i)}] = S_0^{(i)} \exp(\mu_i), \\ V_{ij}^{BS} &= Cov[S_1^{(i)}, S_1^{(j)}] = E_i^{BS} E_j^{BS} \{\exp(\rho_{ij}^{BS} \sigma_i \sigma_j) - 1\}, \quad i, j = 1,2. \end{aligned} \tag{6.10}$$

For illustration set $S_0^{(k)} = 1,\ \mu_k = 0,\ k = 1,2$. Applying the Black-Scholes deflator from Theorem 3.1 one obtains Margrabe's classical exchange option price at initial time $t=0$ (e.g. [3], Theorem 2)

$$M^{BS} = E[D_1(S_1^{(1)} - S_1^{(2)})_+] = 2 \cdot \Phi(\tfrac{1}{2}\sigma) - 1, \quad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho^{BS}\sigma_1\sigma_2}\,. \tag{6.11}$$

In the bivariate variance-Erlang model (with closed-form exchange option price) the future prices at unit time $t=1$ are described by equations of the form (5.1) such that

$$S_1^{(k)} = \exp(\xi + X_k), \quad X_k = \theta_k W + \tau_k \sqrt{W} Z_k, \quad k = 1,2, \tag{6.12}$$

with $W \sim Erlang(m)$, and the Z_k's are correlated standard normal with correlation coefficient ρ^{VE}. The corresponding means and variance-covariance characteristics are

$$\begin{aligned} E_i^{VE} &= E[S_1^{(i)}] = \exp\{\xi - m \cdot \ln(1-\alpha_i)\}, \quad \alpha_i = \theta_i + \tfrac{1}{2}\tau_i^2, \\ V_{ij}^{VE} &= Cov[S_1^{(i)}, S_1^{(j)}] = \exp\{2\xi - m \cdot \ln(1 - \alpha_i - \alpha_j - 2\rho_{ij}^{Ve}\tau_i\tau_j\} - E_i^{BS} E_j^{BS},\ i, j = 1,2. \end{aligned} \tag{6.13}$$

It is natural to compare both models under equal first and second order moments. In case $S_0^{(k)} = 1,\ \mu_k = 0,\ k = 1,2$, the equations of equal mean and variance-covariance, which consists to equate (6.10) and (6.13), result into the following parameter constraints

$$\begin{aligned} &\xi = m \cdot \ln(1-\alpha_1), \quad \alpha_2 = \alpha_1 = \theta_1 + \tfrac{1}{2}\tau_1^2, \quad \theta_2 - \theta_1 = \tfrac{1}{2}(\tau_1^2 - \tau_2^2), \\ &\sigma_k^2 = m \cdot \ln\left(\frac{(1-\alpha_1)^2}{1 - 2\alpha_1 - \tau_k^2}\right), \quad \rho^{BS}\sigma_1\sigma_2 = m \cdot \ln\left(\frac{(1-\alpha_1)^2}{1 - 2\alpha_1 - \rho^{VE}\tau_1\tau_2}\right). \end{aligned} \tag{6.14}$$

Calculation of the deflated exchange option price (6.8) requires the parameters of the VG deflator in (4.23), which are herewith determined by

$$\alpha = r + C_{(X_1,X_2)}(-\beta), \quad \beta_k \tau_k^2 = \theta_k + \tfrac{1}{2}\tau_k^2 + \rho^{VE}\tau_1\tau_2, \quad k = 1,2. \tag{6.15}$$

For the bivariate symmetric Laplace special case $m=1$ with the simple parameter choice

$$\rho^{VE} = 0, \quad \theta_2 = \theta_1 = 0, \quad \tau_2 = \tau_1 = \tau, \text{ one obtains} \tag{6.16}$$

$$\alpha = r + C_{(X_1,X_2)}(-\beta) = r - \ln(1 - \tfrac{1}{4}\tau^2), \quad \beta_1 = \beta_2 = \tfrac{1}{2}. \tag{6.17}$$

The remaining parameters in (6.8) take the values

$$a_2 = a_1 = \tfrac{1}{4}\tau^2, \quad b_1 = \tfrac{\sqrt{2}}{2}\tau, \quad b_2 = -\tfrac{\sqrt{2}}{2}\tau. \tag{6.18}$$

A straightforward calculation of (6.8) using (5.8) and (5.10) for $m=1$ yields the symmetric Laplace Margrabe option price

$$M^{SL} = E[D_1(S_1^{(1)} - S_1^{(2)})_+] = e^{-r} \cdot (1 - \tfrac{1}{2}\tau^2) \cdot \frac{\tau}{2}. \tag{6.19}$$

Numerically, if $r=0, \tau=0.1$, then $\sigma=0.14249$, and $M^{BS}=5.67991\%$ compares with $M^{SL}=4.975\%$. The follow-up [8] provides some comparisons based on stock market indices.

7. RESULTS, FURTHER DISCUSSION AND CONCLUSIONS

First, it is useful to summarize what has been accomplished. The starting point is the wish to enrich multidimensional option pricing with some multivariate non-Gaussian models able to capture skewness, kurtosis and other stylized facts from observed insurance liability and financial market data. It turns out that not much is known in this respect, especially when it comes to market consistent valuation for use in the Basel III and Solvency II projects. To achieve this, one can either work with an equivalent martingale measure for deflated price processes or use a state-price deflator to discount insurance liabilities and asset prices under the real-world probability measure. The second possible path has been followed. Since a multivariate exponential variance-gamma process is one of the popular alternative choices to the multivariate exponential Gaussian process, the construction of an explicit state-price deflator for it is a main first goal, which has been achieved in Theorem 4.1. An application to bivariate option pricing has been undertaken. Theorem 6.1, which displays a Margrabe formula for the deflated exponential shifted bivariate variance-gamma process, is a main new contribution. Moreover, analytical closed-form expressions for the bivariate exponential variance-Erlang special case are also obtained.

Second, it is important to remark that our simple model is easy to work with but has some serious drawbacks. For example, linear correlation cannot be fitted once the margins are fixed. Moreover, the choice of a single parameter ν causes great difficulty in the joint calibration to option prices on the margins, as observed in [21]. To overcome these difficulties there are at least two possibilities at disposal. One can either apply an alternative estimation method, which does not share some of the inconveniences (main purpose of the follow-up [8]), or consider more complex multivariate variance-gamma models. Results on the latter proposal will be presented elsewhere.

APPENDIX 1: *Integral identities of normal type*

The crucial identities used in the derivation of Theorem 5.1 are stated and proved separately.

Lemma A1.1 For any real numbers b,c,μ and $\sigma>0$ one has the identity

$$\sigma^{-1}\cdot\int_c^{\infty} e^{bx}\cdot\varphi((x-\mu)/\sigma)dx = e^{b\mu+\frac{1}{2}b^2\sigma^2}\cdot\Phi\left(\tfrac{\mu-c}{\sigma}+b\sigma\right). \quad \text{(A1.1)}$$

Proof Consider first the case $\mu=0,\sigma=1$. From the relation $e^{bx}\varphi(x)=e^{\frac{1}{2}b^2}\varphi(x-b)$ one gets

$$\int_c^{\infty} e^{bx}\cdot\varphi(x)dx = e^{\frac{1}{2}b^2}\cdot\int_{c-b}^{\infty}\varphi(t)dt = e^{\frac{1}{2}b^2}\cdot\Phi(b-c).$$

Using this one obtains by a change of variables

$$\sigma^{-1}\cdot\int_c^{\infty} e^{bx}\cdot\varphi((x-\mu)/\sigma)dx = \int_{(c-\mu)/\sigma}^{\infty} e^{b\mu+b\sigma t}\cdot\varphi(t)dt = e^{b\mu+\frac{1}{2}b^2\sigma^2}\cdot\Phi\left(\tfrac{\mu-c}{\sigma}+b\sigma\right). \ \lozenge$$

Lemma A1.2 For any real numbers a,b,μ and $\sigma>0$ one has the identity

$$\sigma^{-1}\cdot\int_{-\infty}^{\infty}\Phi(a+bx)\varphi((x-\mu)/\sigma)dx = \Phi\left(\tfrac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right). \quad \text{(A1.2)}$$

Proof Consider the functions $F(z)=\int_{-\infty}^{\infty}\Phi(z+x)\varphi(x)dx,\quad G_a(z)=\int_{-\infty}^{\infty}\Phi(a+zx)\varphi(x)dx$. One notes that $F(0)=\int_{-\infty}^{\infty}\Phi(x)\varphi(x)dx=\frac{1}{2}$ and $F'(z)=\int_{-\infty}^{\infty}\varphi(z+x)\varphi(x)dx=\frac{\sqrt{2}}{2}\varphi(\frac{z}{\sqrt{2}})$, from which one gets $F(a)=F(0)+\int_0^a F'(z)dz=\Phi(\frac{a}{\sqrt{2}})$. On the other hand, one has $G_a(1)=F(a)=\Phi(\frac{a}{\sqrt{2}})$ and

$G_a^{'}(z) = z \cdot \int_{-\infty}^{\infty} \varphi(a+zx)\varphi(x)dx = \frac{z}{(1+z^2)^{3/2}}\varphi(\frac{a}{\sqrt{1+z^2}})$, hence $G_a(b) = G_a(1) + \int_1^b G'(z)dz = \Phi(\frac{a}{\sqrt{1+b^2}})$. It follows that $\sigma^{-1} \cdot \int_{-\infty}^{\infty} \Phi(a+bx)\varphi((x-\mu)/\sigma)dx = G_{a+b\mu}(b\sigma) = \Phi\left(\frac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right)$. ◊

APPENDIX 2: *Closed-form evaluation of an integral function of normal type*

In case $\gamma = m = 1,2,...$, is an integer parameter, the evaluation of the integral (5.9) is done through partial integration by finding first a primitive integral for $x^{2m-1}\varphi(x)$ of the form

$$\int x^{2m-1}\varphi(x)dx = -P_m(x)\cdot\varphi(x), \qquad \text{(A2.1)}$$

where $P_m(x)$ is a polynomial of degree $2(m-1)$ with integer coefficients. The following recursive relationship determines this polynomial completely.

Lemma A2.1 The polynomial $P_m(x)$ defined by (A2.1) satisfies the following recursion

$$P_m(x) = x^{2(m-1)} + 2(m-1)P_{m-1}(x), \quad m = 2,3,..., \quad P_1(x) = 1. \qquad \text{(A2.2)}$$

Proof First of all, since $\varphi'(x) = -x\varphi(x)$ it is immediate that $-\varphi(x)$ is a primitive integral of $x\varphi(x)$, hence $P_1(x) = 1$. For $m \ge 2$ the recursion is shown by induction. For $m = 2$ one proceeds as follows. Let $H_k(x)$ be the (probabilistic) Hermite polynomial of order k, $k = 0,1,2,...$, which satisfies the property $\varphi^{(k)}(x) = (-1)^k H_k(x)\varphi(x)$. In particular, one has

$\varphi^{(3)}(x) = -(x^3 - 3x)\varphi(x)$, hence $x^3\varphi(x) = 3x\varphi(x) - \varphi^{(3)}(x) = -3\varphi'(x) - \varphi^{(3)}(x)$.

This leads to the primitive integral

$\int x^3\varphi(x)dx = -3\varphi(x) - \varphi''(x) = -(3 + H_2(x))\varphi(x) = -(x^2+2)\varphi(x)$, hence

$P_2(x) = x^2 + 2 = x^2 + 2P_1(x)$,

which shows (A2.2) for $m = 2$. The induction step from m to $m+1$ is shown by performing two successive partial integrations as follows:

$$\begin{aligned}
&\int x^{2m+1}\varphi(x)dx = \int [x^{2m-1}\varphi(x)]\cdot x^2 dx = x^2 \cdot \int x^{2m-1}\varphi(x)dx - 2\int x \cdot [\int x^{2m-1}\varphi(x)dx]dx \\
&= x^2 \cdot \left(-x^{2(m-1)}\varphi(x) + 2(m-1)\int x^{2m-3}\varphi(x)dx\right) - 2\int[-x^{2m-1}\varphi(x) + 2(m-1)x\cdot\int x^{2m-3}\varphi(x)dx]dx \\
&= -x^{2m}\varphi(x) + 2(m-1)x^2 \cdot \int x^{2m-3}\varphi(x)dx \\
&+ 2\cdot\int x^{2m-1}\varphi(x)dx - 2(m-1)x^2 \cdot \int x^{2m-3}\varphi(x)dx + 2(m-1)\cdot\int x^{2m-1}\varphi(x)dx \\
&= -x^{2m}\varphi(x) + 2m\cdot\int x^{2m-1}\varphi(x)dx,
\end{aligned}$$

which shows the recursion for the index $m+1$, namely $P_{m+1}(x) = x^{2m)} + 2mP_m(x)$. ◊

A successive application of the recursion (A2.2) yields the following explicit formula.

Lemma A2.2 The polynomial $P_m(x)$ satisfies the explicit representation

$$P_m(x) = 2^{m-1}(m-1)!\sum_{k=0}^{m-1}\frac{1}{k!}\left(\frac{x^2}{2}\right)^k, \quad m = 1,2,3,.... \qquad \text{(A2.3)}$$

Proof From Lemma A2.1 one knows that $P_m(x)$ is a polynomial in even powers of x, say $P_m(x) = \sum_{k=0}^{m-1} a_{m,k}x^{2k}$. With (A2.2) one obtains the recursive relationship

$a_{m,k} = 2(m-1)\cdot a_{m-1,k}, \quad k = 0,1,...,m-2, \quad a_{m,m-1} = 1, \quad m = 2,3,4,...,$

which implies that $a_{m,k} = 2^{m-1}(m-1)!\frac{1}{2^k k!}, \quad k = 0,1,\dots,m-1, \quad m = 1,2,3,\dots$ ◊

Now, let us derive the formula (5.10). A partial integration of (5.9) using (A2.1) yields

$$L_m(d) = \sqrt{2\pi} \cdot \int_0^\infty x^{2m-1}\varphi(x)\Phi(dx)dx = \sqrt{2\pi} \cdot \left(-P_m(x) \cdot \varphi(x) \cdot \Phi(dx)\Big|_0^\infty + d \cdot \int_0^\infty P_m(x)\varphi(x)\varphi(dx)dx \right)$$

$$= P_m(0) \cdot \tfrac{1}{2} + d \cdot \int_0^\infty P_m(x)\varphi(\sqrt{1+d^2}\,x)dx = \frac{1}{2} \cdot \left(P_m(0) + \frac{d}{\sqrt{1+d^2}} \cdot \int_0^\infty P_m(t/\sqrt{1+d^2})2\varphi(t)dt \right).$$

Using (A2.3) it follows that

$$L_m(d) = 2^{m-2}(m-1)!\left(1 + \frac{d}{\sqrt{1+d^2}}(1 + \sum_{k=1}^{m-1} \frac{1}{2^k k!(1+d^2)^k} \int_0^\infty t^{2k} 2\varphi(t)dt) \right), \quad m = 1,2,\dots.$$

Next, one observes that the function $2\varphi(x), 0 \le x < \infty$, is the probability density of the half-normal distribution with moment generating function (e.g. [36], equation (9))

$m(t) = 2\exp(\tfrac{1}{2}t^2)\Phi(t)$.

Through induction one shows the recursive relation

$$m^{(k)}(t) = (k-1) \cdot m^{(k-2)}(t) + t \cdot m^{(k-1)}(t), \quad k = 2,3,\dots,$$

which shows that the moments m_k of the half-normal distribution satisfy the recursion $m_k = (k-1)m_{k-2}$, $k = 2,3,\dots$. Since $m_0 = 1$ one sees that

$$\int_0^\infty t^{2k} 2\varphi(t)dt = m_{2k} = (2k-1)!! = \sum_{j=1}^{k}(2j-1), \quad k = 1,2,3,\dots,$$

The formula (5.10) is shown. ◊

REFERENCES

[1] Milterssen K.R. and Persson S.A., Pricing rate of return guarantees in a Heath-Jarrow-Morton framework, Insurance Math. Econom. 25(3), 307-325 (1999).

[2] Jeanblanc M., Yor M. and Chesney M., Mathematical Methods for Financial Markets, Springer Finance, Springer-Verlag London Limited (2009).

[3] Hürlimann W., An extension of the Black-Scholes and Margrabe formulas to a multiple risk economy, Appl. Math. 2(4), 137-142 (2011).

[4] Hürlimann W., The algebra of option pricing: theory and application, Int. J. of Algebra and Statistics 1(2), 68-88 (2012).

[5] Hürlimann W., Option pricing in the multivariate Black-Scholes market with Vasicek interest rates, Mathematical Finance Letters 2(1), 1-18 (2013).

[6] Cont R. and Tankov P., Financial Modelling with Jump Processes, CRC Financial Mathematics Series, Chapman and Hall (2004).

[7] Luciano E. and Schoutens W., A multivariate jump-driven financial asset model, Quantitative Finance 6(5), 385-402 (2006).

[8] Hürlimann W., Margrabe formulas for a simple bivariate exponential variance-gamma price process (II) Statistical estimation and application, Appears in IJSIMR (2013).

[9] Madan D. and Seneta, E., The variance gamma model for share market returns, Journal of Business 63, 511-524 (1990).

[10] Madan D. and Milne F., Option pricing with VG martingale components, Mathematical Finance 1(4), 39-55 (1991).

[11] Madan D., Carr P. and Chang E., The variance gamma process and option pricing. European Finance Review 2, 79-105 (1998).

[12] Madan D., Purely discontinuous asset pricing processes, In: Jouini E., Cvitanic J. and Musiela M. (Eds.), Option Pricing, Interest Rates and Risk Management, Cambridge University Press, Cambridge, 105-153 (2001).

[13] Carr P., Geman H., Madan D.B. and Yor M., The fine structure of asset returns: an empirical investigation, Journal of Business 75(2), 305-332 (2002).

[14] Geman H., Pure jump Lévy processes for asset price modeling, J. of Banking and Finance 26, 1297-1316 (2002).

[15] Küchler U. and Tappe S., Bilateral gamma distributions and processes in financial Mathematics, Stochastic Process. Appl. 118(2), 261-283 (2008).

[16] Küchler U. and Tappe S., Option pricing in bilateral gamma stock models, Statistics and Decisions 27, 281-307 (2009).

[17] Küchler U. and Tappe S., Tempered stable distributions and applications to financial mathematics, Preprint (2011),
URL: http://www.stochastik.uni-hannover.de/fileadmin/institut/pdf/TS_111114.pdf

[18] Kaishev V.K., Stochastic process induces by Dirichlet (B-) splines: modelling multivariate asset price dynamics, Actuarial Research paper no. 195, Cass Business School, London (2010).

[19] Bellini F. and Mercuri L., Option pricing in a conditional bilateral gamma model, Preprint (2010), URL: http://ssrn.com/abstract=2046478

[20] Semeraro P., A multivariate variance gamma model for financial application, Int. J. of Theoretical and Applied Finance 11(1), 1-18 (2008).

[21] Luciano E. and Semeraro P., Extending time-changed Lévy asset models through multivariate subordinators, Collegio Carlo Alberto Working Paper Series, no. 42 (2007).

[22] Luciano E. and Semeraro P., Multivariate time changes for Lévy asset models: characterization and calibration, J. Computational Appl. Math. 233, 1937-1953 (2010).

[23] Wang J., The multivariate variance gamma process and its applications in multi-asset option pricing, Ph.D. Thesis, University of Maryland, College Park (2009).

[24] Hitaj A. and Mercuri L., Portfolio allocation using multivariate variance gamma, Capital Markets: Asset Pricing & Valuation eJournal 4(4), (2012).

[25] Hürlimann W., A moment method for the multivariate asymmetric Laplace distribution, Statistics & Probability Letters 83(4), 1247-1253 (2013).

[26] Kotz S., Kozubowski T.J. and Podgorski K., The Laplace distribution and generalizations: a revisit with applications, Birkhäuser, Boston (2001).

[27] Kotz S., Kozubowski T.J. and Podgorski K., An asymmetric multivariate Laplace Distribution, TR 367, Dept. Stat. Appl. Probab., Univ. California at Santa Barbara (2003).

[28] Visk H., On the parameter estimation of the asymmetric multivariate Laplace Distribution, Comm. Statist. – Theory and Methods 38(4), 461-470 (2009).

[29] Gerber H.U. and Shiu E.S.W., Option pricing by Esscher transforms (with Discussions), Transactions of the Society of Actuaries 46, 99-191 (1994).

[30] Wüthrich M.V., Bühlmann H. and Furrer, H., Market-Consistent Actuarial Valuation, EAA Lecture Notes, vol. 1 (2nd revised, and enlarged edition), Springer-Verlag (2010).

[31] Wüthrich M.V. and Merz M., Financial Modeling, Actuarial Valuation and Solvency in Insurance, Springer Finance (2013).

[32] Bühlmann H., Life insurance with stochastic interest rates, In: G. Ottaviani (Ed.), Financial Risk in Insurance, Springer (1995).

[33] Hürlimann W., An elementary unified approach to loss variance bounds, Bulletin of the Swiss Association of Actuaries, 73-88 (1997).

[34] Hürlimann W., Extremal Moment Methods and Stochastic Orders – Application in Actuarial Science, Research Monograph, 234p, In: BAMV - Boletín de la Asociación Matemática Venezolana, Volume XV, num.1 & num. 2 (2008).

[35] Margrabe W., The value of an option to exchange one asset for another, Journal of Finance 33, 177-186 (1978).

[36] Pourahmadi M., Construction of skew-normal random variables: are they linear combinations of normal and half-normal? J. Statist. Th. And Appl. 3, 314-328 (2007).

AUTHOR'S BIOGRAPHY

Werner Hürlimann has studied mathematics and physics at ETHZ, where he obtained his PhD in 1980 with a thesis in higher algebra. After postdoctoral fellowships at Yale University and at the Max Planck Institute in Bonn, he became 1984 an actuary at Winterthur Life and Pensions, a senior actuary at Aon Re and IRMG Switzerland 2003-06, a senior consultant at IRIS in Zürich 2006-2008, and is currently employed at FRSGlobal Switzerland (a Wolters Kluwer Company). He has been visiting associate professor in actuarial science at the University of Toronto during the academic year 1988-89. More information is found at his homepage https://sites.google.com/site/whurlimann/.