Algorithms for Computing LCA in Complete Binary Trees

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Abstract: Based on properties of the lowest common ancestor (LCA) of two nodes in a complete binary tree, the paper designs three algorithms as well as their C-language codes to compute the LCAs. The designed algorithms contain both an elementary version that is simple, easily-understood and easily-realized and a high-efficient version that can be applied for developments of embedded systems and SoC. The paper is of a reference to development of embedded systems and related area.

Keywords: lowest common ancestor, Algorithm, Embed System, Complete Binary tree.

1. INTRODUCTION

Problem of seeking a lowest common ancestor (LCA) of nodes in a tree was first raised by Alfred Aho and John Hopcroft in 1973. As a basic and common problem in both graph theory and computer science, the problem has been paid attention to because the solution of the problem is helpful to other related problems, as introduced in [1] and [2]. Recent years, complete binary trees that are widely applied in bioinformatics [3], rapid location of data in industrial control [4-6] bring new values for developing algorithms of the problem. The bibliography [7] presents an algorithm for computing an LCA of two neighboring nodes in a complete binary tree, however algorithm for computing that in an arbitrary a complete binary has not been seen.

A recent bibliography [8] has systematically presented properties of LCAs in a complete binary, but the paper does not present an algorithm for the related computations. Therefore, we present related algorithms for it. This paper designs algorithms and C language implementation of computing LCA of nodes in a complete binary tree. The paper also makes an analysis of time-complexity for the algorithms.

2. PRELIMINARIES

This section first presents preliminaries for later sections.

2.1 Basic Definitions and Symbols

Symbol \( \lfloor x \rfloor \) denotes a floor function of a real number such that satisfies \( \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \). Symbol \( \{ x \} \) is the decimal function such that \( x = \lfloor x \rfloor + \{ x \} \) and symbol \( \lceil x \rceil \) is the ceil function that holds \( x \leq \lceil x \rceil < x + 1 \). Symbol \( N_{d,j} \) is to express the node at the \( j \)-th position on the \( k \)-th level in a binary tree, symbol \( G_{d,j}^{(i,j)} \) is to express LCA if \( N_{d,j} \) and \( N_{d,j} \). The whole paper suggests the maximal depth of the binary tree is \( h \).

2.2 Lemmas

Lemma 1 ([9]) For arbitrary real \( \delta > 0 \), integer \( J = \lceil \log_2 \delta \rceil + 1 \) is the minimal \( j \) that fits the equation \( \frac{\delta}{2^j} = 0 \). 

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Lemma 2 ([9]) For arbitrary real $\alpha, \delta > 0$ and positive integer $j$, the condition $\left[ \frac{\alpha + \delta}{2^j} \right] = 0$ is necessary for the equation $\left[ \frac{\alpha + \delta}{2^j} \right] = \left[ \frac{\alpha}{2^j} \right]$. If a positive integer $J$ is the minimal $j$ that holds $\left[ \frac{\delta}{2^j} \right] = 0$, then $\left[ \frac{\alpha + \delta}{2^j} \right] = \left[ \frac{\alpha}{2^j} \right] = 0$.

Lemma 3 ([10]) For arbitrary positive integer $\delta$ and non-negative $I$, the inequality $\left( \frac{1}{2} \right) \times 2^I < \delta \leq \left( \frac{1}{2} \right) \times 2^{I+1}$ has a unique solution $\chi = \left[ \frac{\delta - \frac{1}{2}}{2^I} \right]$.

Lemma 4 ([8]) Let $N_{(i, \alpha)}$ and $N_{(i, \beta)}$ ($1 < k < 2^{i-1} \leq \alpha < \beta < 2^i$) be two nodes on the $k$-th level in a complete binary tree; if $I$ is the smallest $i$ that fits the equation $\left[ \frac{\alpha}{2^i} \right] = \left[ \frac{\beta}{2^i} \right]$ and $\sigma = \left[ \frac{\alpha}{2^i} \right]$; then the LCA of $N_{(i, \alpha)}$ and $N_{(i, \beta)}$.

Lemma 5 ([8]) Let $N_{(i, \alpha)}$ and $N_{(i, \beta)}$ ($1 < k < 2^{i-1} \leq \alpha < \beta < 2^i$) be two nodes on the $k$-th level in a complete binary tree; if $I$ is the smallest $i$ that fits $0 \leq \alpha \mod 2^I < 2^{i-1}$, integers $\alpha$ and $\chi$ are such that $\chi = \left[ \frac{\alpha}{2^i} \right]$ and $\left( \frac{1}{2} \right) \times 2^I < \beta - \alpha \leq \left( \frac{1}{2} \right) \times 2^{I+1}$, then $G_{(i, \alpha)}^{(i, \beta)} = G_{(i, \beta)}^{(i, \alpha)}$.

Lemma 6 ([8]) Let $N_{(i, \alpha)}$ and $N_{(i, \beta)}$ ($1 < k < 2^{i-1} \leq \alpha < \beta < 2^i$) be two nodes on the $k$-th level in a complete binary tree; if $I$ is the smallest $i$ that fits $0 \leq \alpha \mod 2^I < 2^{i-1}$ and $\sigma = \left[ \frac{\alpha}{2^i} \right]$, $\left[ \frac{\beta}{2^i} \right]$, then the LCA of $N_{(i, \alpha)}$ and $N_{(i, \beta)}$.

Lemma 7 ([8]) For valid number $(k, j)$, if $I$ is the smallest $i$ that fits $0 \leq \alpha \mod 2^I < 2^{i-1}$ and $\sigma = \left[ \frac{\alpha}{2^i} \right]$, $\left[ \frac{\beta}{2^i} \right]$, then the LCA of $N_{(i, \alpha)}$ and $N_{(i, \beta)}$.

3. ALGORITHM DESIGN AND ANALYSIS

This section design concrete algorithms according to previous lemmas. The algorithms can be applied for arbitrary nodes $N_{(i, \alpha)}$ and $N_{(i, \beta)}$ in a complete binary tree, where $\alpha$ and $\beta$ are sequential indices and $\alpha < \beta$.

3.1 Algorithm Design

3.1.1 Algorithm I Based on Lemma 4

The algorithm is by 4 steps as follows:

i. Compute $m, n$ by $\alpha$ and $\beta$ respectively, $m = \left\lfloor \log_2 \alpha \right\rfloor + 1$, $n = \left\lfloor \log_2 \beta \right\rfloor + 1$;

ii. Compute $\sigma = |m - n|, \gamma = \left\lfloor \frac{\beta}{2^m} \right\rfloor$;

iii. Compute $I$ by the following loop

\[ I=1; \; \sigma_1 = \left\lfloor \frac{\alpha}{2^m} \right\rfloor, \sigma_2 = \left\lfloor \frac{\gamma}{2^n} \right\rfloor; \]

While ($\sigma_1 \neq \sigma_2$)

Begin

\[ I=I+1; \; \sigma_1 = \left\lfloor \frac{\alpha}{2^m} \right\rfloor, \sigma_2 = \left\lfloor \frac{\gamma}{2^n} \right\rfloor; \]

End

iv. Compute LCA= $N_{(m, I, \sigma_1)}$. 

The C++ language for the algorithm I is as follows

```cpp
int FindLCA(int a, int b) {
    int t = 1, temp1 = a, temp2 = b, n, m, I;
    m = (int)(log10(i)/log10(2)); /*compute m*/
    n = (int)(log10(j)/log10(2)); /*compute n*/
    I = n - m; t <<= I;
    temp2 = (int)floor(j/t);
    for (int r = 1;; r++) {
        temp1 >>= 1; temp2 >>= 1;
        if (temp1 == temp2) return temp2;
    }
}
```

3.1.2 Algorithm II Based on Lemma 5 and 6

The algorithm is by 6 steps as follows:

i. Compute \( m, n \) by \( \alpha \) and \( \beta \) respectively; \( m = \lfloor \log_2 \alpha \rfloor + 1, n = \lfloor \log_2 \beta \rfloor + 1 \)

ii. Compute \( \sigma = n - m, \gamma = \lfloor \frac{\beta}{2^2} \rfloor \);

iii. Compute \( t = \min(i | 0 \leq \alpha \mod 2^i < 2^{i+1}) \);

iv. Compute \( \delta = |\gamma - \alpha|, \sigma = \lfloor \frac{\alpha}{2^2} \rfloor \);

v. Compute non-negative integer \( \chi \): \( \chi = \left\lfloor \frac{\delta}{2^2} - \frac{1}{2} \right\rfloor \);

vi. If \( \chi = 0 \), then \( N_{\sigma - I, \sigma} \) is the LCA, or if \( \alpha = \sigma \), then \( \gamma = \sigma + \chi \) and goto step iii.

The C++ language for the algorithm II is as follows:

```cpp
int FindLCA(int a, int b) {
    int I, _2i = 1, n, m, r, delta, rm;
    int x = b - a;
    while (x != 0) {
        a = (int)floor(a / ((double)_2i)); r = a + x;
        m = (int)(log10(a)/log10(2));
        n = (int)(log10(r)/log10(2));
        I = n - m; _2i = 1; _2i <<= I;
        r = (int)floor(r / ((double)_2i));
        if (r == a) return a / 2;
        else if (r < a) Swap(&a, &r);
        _2i = 1;
        for (int i = 1; i++ ) {
            _2i <<= 1; rm = a & (_2i - 1);
            if (rm < _2i) { I = i; break; }
        }
        delta = abs(r - a); _2i = 1; _2i <<= I;
    }
}
x=(int)ceil(delta/((double)_2i)-0.5);
return (int)floor(a/((double)_2i));

3.1.3 Algorithm III Based on Lemma 1, 2, 6 and 7

The algorithm is by 6 steps as follows:

i. Compute \( m, n \) by \( \alpha \) and \( \beta \) respectively, \( m=\lceil \log_2 \alpha \rceil+1, \quad n=\lceil \log_2 \beta \rceil+1; \)

ii. Compute \( \sigma = n-m, \quad \gamma = \left| \frac{\beta}{2^{\sigma}} \right|; \)

iii. Compute \( \delta = |\gamma - \alpha|; \)

iv. Compute \( I_1 = \lceil \log_2 \delta \rceil +1, \quad \sigma_1 = \left\lfloor \frac{\alpha}{2^{\sigma}} \right\rfloor, \sigma_2 = \left\lfloor \frac{\gamma}{2^{\sigma}} \right\rfloor; \)

v. If \( \sigma_1 = \sigma_2 \), then \( N_{(m-I_1, \alpha)} \) is what we need; Or if \( \alpha \leq \frac{\sigma_1}{2^{\gamma}} \), then goto step vi;

vi. Compute \( I_i = \min(i)(0 \leq \alpha \bmod 2^i < 2^{-i}) \), Compute \( \sigma = \left\lfloor \frac{\alpha}{2^{\gamma}} \right\rfloor; \) then \( N_{(m-I_i, \alpha)} \) is LCA

The C++ language for the algorithm III is as follows:

```cpp
int FindLCA(int a, int b)
{
    int I1, I2, _2i=1, n, m, r, rm;
    int sigama1, sigama2, sigama, delta;
    m= (int)(log10(a)/log10(2));
    n= (int)(log10(b)/log10(2));
    I = n-m;  _2i=1;  _2i<<=I;
    r=(int)floor(b/((double)_2i));
    if (r==a) return a/2; else if (r<a) Swap(&a,&r);
    delta=r-a;
    I1=(int)floor(log10(delta))/log10(2))+1;
    _2i=1;  _2i<<=I1;
    sigama1=(int)floor(a/((double)_2i));
    sigama2=(int)floor(r/((double)_2i));
    if(sigama1==sigama2) return sigama2;
    a=sigama1;  _2i = 1;
    for(int i=1;i++)
    {  
        _2i<<= 1;  rm=a&(_2i - 1);
        if(rm<_2i/2) {I2=i;  break;  }
    }
    sigama=(int)floor(a/((double)_2i));
    return sigama;
}
```

3.2 Analysis of Algorithms

The step iii in Algorithm I is a loop that wastes most time. By Lemma 4, the number of loops depends on the number of level \( m \). Hence it at most needs \( m \) loops. Consequently, the time complexity of Algorithm I is \( T_I(m) = m = O(m) \).

Algorithm II is a bi-looped one. Its inner loop is to search the minimal \( I \) that fits \( 0 \leq \alpha \bmod 2^i < 2^{-i} \). By Lemma 1, this needs \( \lceil \log_2 \alpha \rceil +1 \) computations. The other loop is from step iii to step vi with
initial condition \( \chi = \delta \) and terminal condition \( \chi = 0 \). In the loop, it requires to divide \( \chi \) by \( 2^r \), which needs at most \( \lfloor \log_2 \delta \rfloor + 1 \) computations. Therefore, the whole time needed is
\[
T_3(\alpha, \delta) = (\lfloor \log_2 \delta \rfloor + 1)(\lfloor \log_2 \alpha \rfloor + 1)
\]
It is easy to prove the worst case is that \( \alpha = \delta \) and this time it holds
\[
\max(T_3(\alpha, \delta)) = (\lfloor \log_2 \alpha \rfloor + 1)^2 \leq (\log_2 \alpha + 1)^2 = O(\log^2 \alpha)
\]
Particularly, when \( \delta = 1 \),
\[
\max(T_3(\alpha, 1)) = \lfloor \log_2 \alpha \rfloor + 1 = O(\log \alpha)
\]
Since in a complete binary tree, the level of node \( \alpha \) lying is \( \lfloor \log_2 \alpha \rfloor + 1 \), hence we have
\[
\max(T_3(\alpha)) = m^2
\]
The algorithm III contain a loop in step vi, which takes at most \( \lfloor \log_2 \left( \frac{\alpha}{2^v} \right) \rfloor + 1 \) computation. Hence the total time needed is
\[
T_3(\alpha) = \left( \lfloor \log_2 \left( \frac{\alpha}{2^v} \right) \rfloor + 1 \right) \leq \log_2 \left( \frac{\alpha}{2^v} \right) + 1 \leq \log_2 \alpha - \log_2 \left( \frac{\alpha}{2^v} \right)
\]
Namely
\[
T_3(\alpha) \leq \log_2 \alpha = O(\log \alpha)
\]

4. CONCLUSION AND FUTURE WORD

Seen from the time complexities, each of the previous designed three algorithms has its own specialty. From point of view of a programmer, the algorithm I is easy to program because it is simple. However, it is a little rough because we can only know its time complexity to its level while the other two can reach to a node.

It is obviously the algorithm II is the lowest efficiency seen from the time complexity. However it is strictly derived from properties of the floor function and it is worth of more digging out. For example, if the step vi is improved by other means, it might be a better one. It remains us a problem whether there is some other better approach to develop.

The algorithm III, we think it the best one because it is a most industrial one. This is because it contains two industrial traits: one is a better time complexity, and the other is its reconstructability.

In fact, the core of the algorithm III is its step vi, namely, computation of \( I_5 = \min(i) \{0 \leq \alpha \mod 2^v < 2^{v - 1} \} \). Note that, nodes in a complete binary tree can be previously coded by the way from top to bottom and from left to right. Hence each node \( \alpha \) must have a minimal \( I \) that fits \( 0 \leq \alpha \mod 2^v < 2^{v - 1} \). If we preprocess and record these Is, it only takes \( O(\log \alpha) \) time. After that preprocess, computation of LCA is turned to be an \( O(1) \) inquiry process, which is quite suit for developing embedded system.

ACKNOWLEDGEMENTS

The research work is supported by the national Ministry of science and technology under project 2013GA780052, Department of Guangdong Science and Technology under projects 2012B011300068, Foshan Bureau of Science and Technology under projects 2013AG10007, Special Innovative Projects from Guangdong Education Department, and Chancheng government under projects 2013B1018 and 2013A1021. The authors sincerely present thanks to them all.

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