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A Note on Delta Hedging of Higher-order Interest Rate Derivatives

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Abstract: *We consider the problem of delta hedging higher-order derivatives with interest rates governed by short rate processes. We investigate the implications of taking the numerical differentiation of the option price with respect to the price of the underlying security to provide the delta-neutral strategy, a procedure which often appears in the literature. We found negatively biased hedging errors from the perspective of the option seller. In this paper, we provide a better delta strategy taking the derivative with respect to the short-term interest rate - the non-tradable asset which affects the whole term structure.*

Keywords: *Dynamic Hedging, Interest Rates Derivatives, Delta Hedging, Computational Finance.*

1. INTRODUCTION

As important as to price financial derivative contracts is to hedge them. Financial institutions and investors want to know how instruments prices behave due to changes in market conditions. Additionally, they need to quantify it and manage the associated risks.

There is a large body of literature examining the pricing tools of various models for different markets. Usually, the issue of managing price risk in options is treated in basic and intermediary books such as [13] and [2] by simply taking the first numerical partial derivative of the contract's price with respect to the price of the underlying asset in continuous-time models. This leads to a delta-neutral strategy, where the position in the underlying asset is continuously rebalanced in order to make the portfolio insensitive to changes in the price of the option.

In fixed income markets, the standard practice is numerical calculating the first partial derivative of the contract's price with respect to the price of a prescribed zero coupon - which stands as the underlying asset (see, e.g, [13]). However, securities in these markets are sensitive to changes in the term structure of the interest rates as a whole, so this approach can actually lead to incorrect results. Our analysis focuses on the impact in the prices of derivatives caused by changes in the whole yield curve. Higher order derivatives, such as bond options and swaptions, are sensitive to these changes, as they depend on more than one point of the curve (in contrast, bonds and caps do not). Such impact of changes becomes even more relevant if we choose partial differential equations as a pricing tool. In fact, different from options subscribed on stocks - where the state variable is the stock price, options subscribed on bonds have the interest rate process as their stochastic variable, which cannot be bought and sold as an asset.

Therefore, to establish the delta hedging strategy for higher order interest rate derivatives, we need to consider the option's sensitivity to all points of the term structure which influences its price. We do this perturbing the underlying stochastic variable that stands for the market's non-tradable asset, namely the short rate process, by which means we build a much more accurate delta-neutral hedging portfolio composed by a quantity of bonds that subscribe the option and money in the money market account. This is different from simply recalculating the first numerical partial derivative of the contract price with respect to the price of the zero-coupon bond, as the numerical results herein strongly suggest. Indeed, a numerical example we performed shows that the difference between our term structure delta hedging strategy and the standard delta hedging procedure may be of the order of 100% better. We consider bond options to build our numerical results. However, they can be easily extended to delta hedge a payer swaption. The construction here is possible even in the absence of closed-form expressions for the price of the derivative.

2. Dynamic Hedging

2.1. A Short Rate Model and the Feynman-Kac Theorem

The economy we consider has the trading interval $[0,S]$. The uncertainty involved in this economy is completely specified by the measurable space $(\Omega,\mathcal{F},\mathbb{F}\equiv(\mathcal{F}_t)_{0\leq t\leq S}),\mathcal{F}_t\subseteq\mathcal{F}$, where Ω denotes the set of all possible outcome elements $\omega\in\Omega$, and $\mathbb{F}$ denotes a filtration containing all relevant information. This space is equipped with a probability measure $\mathbb{P}$ reflecting the real world probability law.

We consider the economy driven by the Vasicek short-term interest rate model [10][1], with the associated diffusion process a mean-reverting version of the Ornstein-Uhlenbeck process. Namely, the short-term interest rate process r_t is defined as the unique strong solution of the stochastic differential equation (SDE)

$$dr_t = a(b-r_t)dt + \sigma dW_t^{\mathbb{Q}} \tag{1}$$

where a,b and σ are strictly positive constants, b designates the mean reversion level, a is the reversion speed and σ is the volatility of the short rate (see, e.g., [8]). Under the probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, the process $W^{\mathbb{Q}}$ is a one-dimensional Brownian motion and the price process of any $\mathcal{F}_T$-measurable derivative in this economy is a Martingale.

In order to price a derivative via PDE, we use

Theorem 1 (Feynman-Kac). Let $T>0, h:\mathbb{R}\to\mathbb{R}$ and

$$U(t,r_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}h(r_T)\mid\mathcal{F}_t\right] \tag{2}$$

where r is the solution of the SDE (1) with initial condition r_0 at $t=0$. Then $([0,T]\times\mathbb{R})\ni(t,x)\mapsto U(t,x)\in\mathbb{R}$ solves

$$\frac{\partial U}{\partial t} + a(b-x)\frac{\partial U}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} = xU \tag{3}$$

with terminal condition

$$U(T,x) = h(x). \tag{4}$$

2.2. Financial Instruments; Pricing Bond Options via PDEs

Pricing derivative instruments of second (or higher) order is computationally much more expensive than pricing those of first order (see order classification in [12]). For instance, the price of a zero-coupon bond option is found firstly by solving the PDE (3) backwards in time from S to T to obtain the price at time T of a zero-coupon bond, where T is the maturity of the option, S is the maturity of the bond and N is the notional value of the contract. Then, using the bond price at time T as the new terminal condition, we solve again (3) in $[0,T]$ to obtain the price of the option at time zero.

Denoting $P(t,\eta)$ the price of the zero coupon bond at time $t\in(0,S)$, where η is the maturity of the bond, the arbitrage free price C_t of a zero-coupon bond call option with strike price K, at time $t\in(0,T), T<S$, is

$$C_t = N\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\max(P(T,S)-K,0)\mid\mathcal{F}_t\right] \tag{5}$$

Under the Vasicek model, a closed-form expression to the above conditional expectation was found in [7][2] :

$$C_t = NP(t,S)\Phi(d_1) - KP(t,T)\Phi(d_2) \tag{6}$$

[1] Our results are easily extensible to other Markovian short rate processes, such as that in [3].

[2] The work in [7] also includes a trick to extend the above result to price options on couponbearing bonds.

Where

$$d_1 = \frac{\ln\left(\frac{NP(t,S)}{KP(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p}$$
$$d_2 = d_1 - \sigma_p \tag{7}$$

and

$$\sigma_p = \sigma\left(\frac{1-e^{-a(S-T)}}{a}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}. \tag{8}$$

The function $\Phi(d)$ corresponds to the probability of a standard normal random variable being less than d, namely

$$\Phi(d) = \int_{-\infty}^{d} \phi(x)dx, \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}. \tag{9}$$

2.3. Delta Sensitivity

The sensitivity of the derivative instrument due to changes in the market price of the underlying is known as delta, defined below[3].

Definition 1. Delta is the infinitesimal change of an instrument's price U given an infinitesimal change in the underlying's price P, all other quantities assumed to be fixed:

$$\Delta = \frac{\partial U}{\partial P}. \tag{10}$$

The approach often used to dynamically hedge a short position is that given by (10). We claim that, in the case of higher-order interest rate derivatives, such approach, in general, will not work, for the main reason that no consideration is taken to the changes that occur in the term structure of interest rates as a whole.

In this section we provide an accurate process to dynamically hedge a short position for a zero-coupon bond option, namely the quantity $\bar{\Delta}$ satisfying

$$dC_t = \bar{\Delta} \cdot dP(t,S) + \frac{\partial C_t}{\partial P(t,T)} \cdot dP(t,T). \tag{11}$$

The above expression captures the sensitivity of the derivative's price with respect to the change in the term structure of interest rates, in which case the representation given via $P(t,S)$ and $P(t,T)$ suffices. The theorem that follows shows that $\bar{\Delta}$ can still be analytically obtained and that calculation is computationally simple.

This approach also serves as a guide for the case of any second order fixed income derivative.

Theorem 2. Let us consider that only parallel shifts in the term structure occurs (e.g., the Vasicek model). Then the quantity $\bar{\Delta}$ of zero-coupon bonds expiring at time S, to be continuously rebalanced in the portfolio that hedges the zero-coupon bond call option, as given in (11), reads as

$$\bar{\Delta} = \frac{\frac{dC_t}{dr} - \frac{dP(t,T)}{dr}\frac{C_t}{P(t,T)}}{\frac{dP(t,S)}{dr}}. \tag{12}$$

Proof. We have that

$$\begin{aligned}
\frac{\partial C_t}{\partial P(t,T)} &= P(T,S)\Phi(d_1) - K\Phi(d_2) + NP(t,S)\frac{\partial \Phi(d_1)}{\partial P(t,T)} \\
&\quad - KP(t,T)\frac{\partial \Phi(d_2)}{\partial P(t,T)} \\
&= P(T,S)\Phi(d_1) - K\Phi(d_2) + NP(t,S)\phi(d_1)\frac{\partial d_1}{\partial P(t,T)} \\
&\quad - KP(t,T)\phi(d_2)\frac{\partial d_2}{\partial P(t,T)}
\end{aligned} \tag{13}$$

[3]An explanation of the static hedging strategy using gamma (the derivative of delta with respect to the underlying) and vega (the derivative of price with respect to the volatility) can be found in [6].

After some algebraic manipulation and reminding that

$$\frac{\partial(d_2)}{\partial P(t,T)} = \frac{\partial(d_1)}{\partial P(t,T)} \tag{14}$$

it follows that

$$\ln\left(\frac{NP(t,S)}{KP(t,T)}\right) = \ln\left(\frac{\phi(d_2)\frac{\partial(d_2)}{\partial P(t,T)}}{\phi(d_1)\frac{\partial(d_1)}{\partial P(t,T)}}\right)$$
$$= \frac{1}{2}\left(d_1^2 - d_2^2\right) = \frac{1}{2}\left(2d_1 - \sigma_p\right)\sigma_p = \ln\left(\frac{NP(t,S)}{KP(t,T)}\right) + \frac{\sigma_p^2}{2} - \frac{\sigma_p^2}{2} \tag{15}$$

Hence,

$$\frac{\partial C_t}{\partial P(t,T)} = P(T,S)\Phi(d_1) - K\Phi(d_2) = \frac{C_t}{P(t,T)} \tag{16}$$

where the last term of (16) is justified by multiplying and dividing the second term by $P(t,T)$. Substitution of (16) in (11) completes the proof.

Expression (12) naturally applies to pricing bond options via PDEs, since sensitivity of the price solution to (3) is with respect to r, not with respect to the zero coupon bond prices.

Expression (12) clearly differs from the numerical derivative of the bond option price with respect to the underlying $P(t,S)$. This naive approach, found e.g. in [2], [11] and [13], disregards the role of the term structure of the interest rates in pricing derivatives, as it picks out only the current underlying price. It actually boils down to an analogous of the Black & Scholes delta ([6]). Namely, we have that(see, e.g., [13], page 66 and page 152):

$$\Delta = \frac{C[P(t,S)+\delta P(t,S)] - C[P(t,S)]}{\delta P(t,S)} \tag{17}$$

Prices of bond options for instance depend on the maturity of the option and the maturity of the underlying bond. When we perturb the underlying bond price as suggested by [13], we are assuming a fixed yield curve in order to evaluate the option's sensitivity due to changes in the short rates. But typically, this is not true. Perturbations in the short rate process, in turn, cause perturbations in the overall yield curve. Both procedures only coincide when one tries to find the delta of a first-order derivative. These aspects are illustrated in Section 3, which shows significative effects in terms of hedging errors. In fact, a conjecture is that the hedging quantity $\bar{\Delta}$ we provide is exact, i.e., the corresponding portfolio produces zero hedging error in the continuous rebalance scheme.

3. Numerical Results

We devised some examples which illustrate the fact that Equation (12) - which stands for the term structure delta, duly captures the sensitivity of zero-coupon bond call options prices with respect to the underlying. In contrast, the naive approach is unable to create a perfect replicating portfolio of the option price.

Figures 1 and 2 respectively stems from (12) and (17), and correspond to a 1-year call option to buy a 2-year zero-coupon bond with strike 0.85 and the following parameters: initial rate $= 0.095$ per year, $a = 0.5 b = 0.1, \sigma = 0.02$. We performed daily rebalances assuming no transaction costs. In figure 2 the lower curve (in red) corresponds to the portfolio value while the upper curve (in blue) corresponds to the option value. In Figure 1 the curves almost coincide. This already suggest a poor tracking of the price under the naive approach and a good one under the new approach.

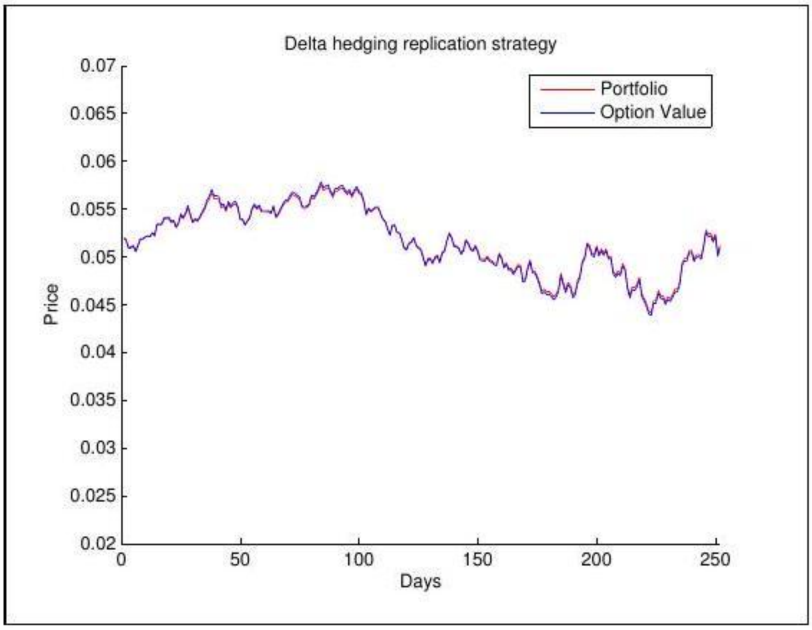

Figure1. *Call option replication with the (correct) term structure delta*

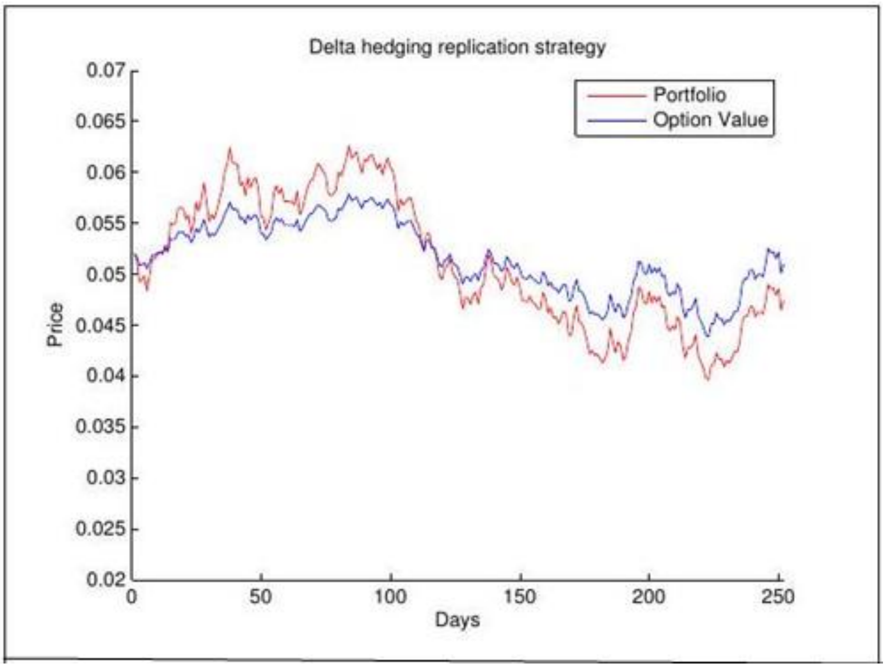

Figure2. *Attempt to replicate call option with the naive delta*

We consider the Vasicek short-term interest rate model to build, as in [4], dynamically a hedging portfolio and check if the delta hedging strategy is in fact able to recast the derivative trading in a zero-sum game. Simulations are performed via an explicit discretization of the stochastic differential equation(1). The antithetic variate technique is used, which aim to reduce the variation of the simulations [5].

The histograms in Figures 3 and 4 were produced generating 10.000 simulation paths of the interest rate, where we consider a daily delta hedge of a zero-coupon bond call option with strike 0.85 and option and bond maturities equal to one and two years, respectively. The interest rate dynamic is the one of figures 1 and 2, i.e., initial interest rate 0.095 per year, $a = 0.5, b = 0.1$ and $\sigma = 0.02$.

The histograms provide the frequency of the hedging error (or hedging cost), a random variable which corresponds to the option value at time of maturity $T(ZCB_{c,T})$ minus the portfolio value also at time of maturity. We denote it e_a in the case of our term structure delta strategy (Figure 3) and e_n in the case of the naive delta strategy (Figure 4).

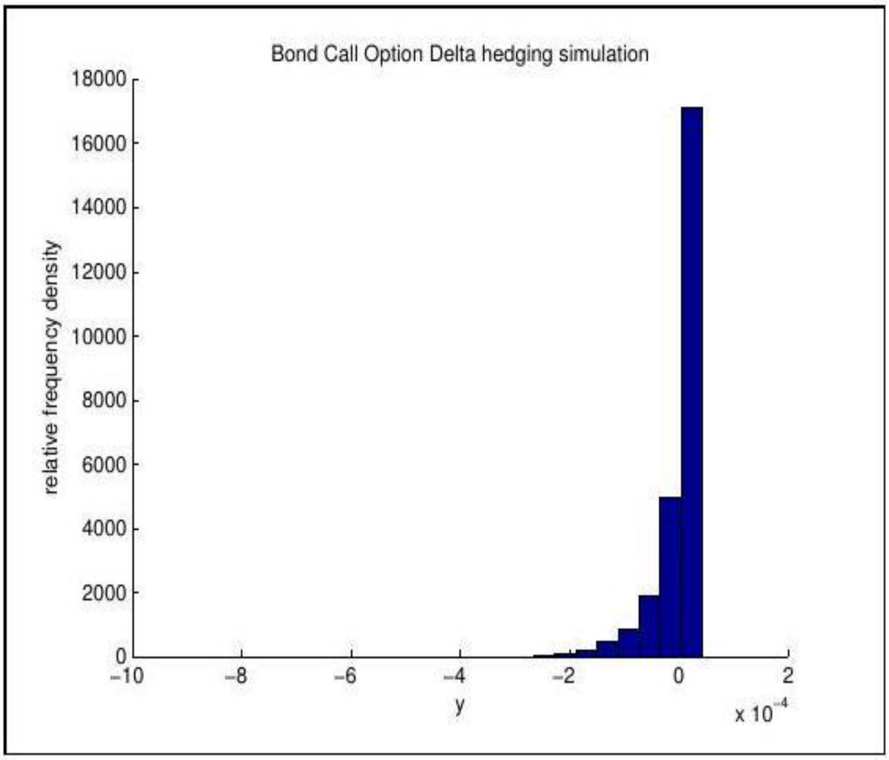

Figure3. *Bond Option term structure delta-hedging histogram*

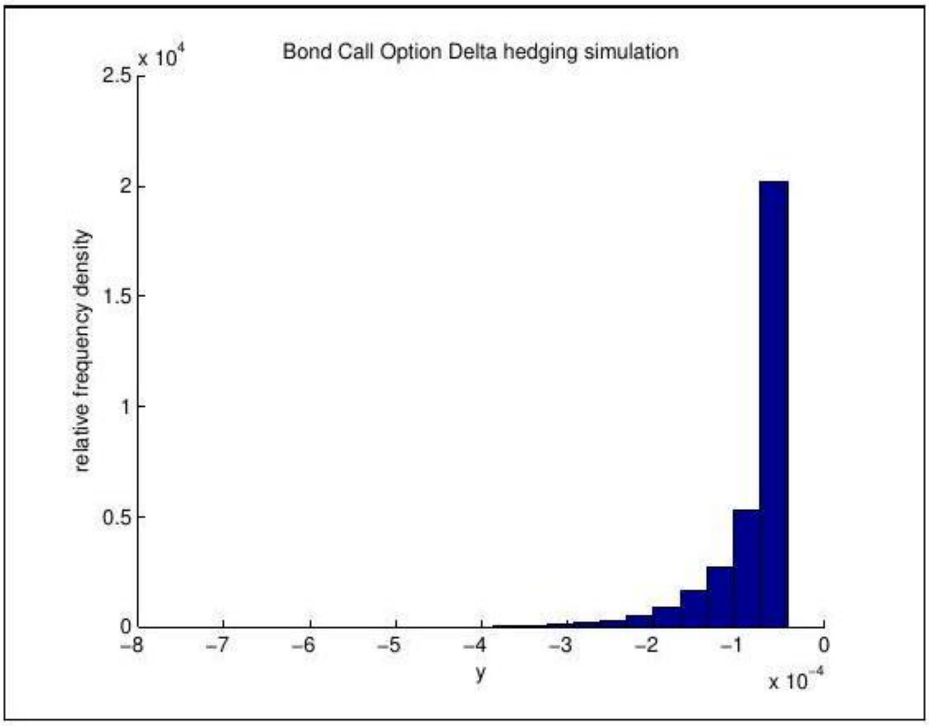

Figure4. *Bond Option naive delta-hedging histogram*

The classical hedging error measure is given by (see, e.g., [1] and the references therein)

$$v_\iota = E[(e_\iota)^2]^{1/2} \tag{18}$$

where ι stands for a or n. In the example, the initial price of each contract is \$0,92 per thousand contracts, and we observe a hedging cost measure of \$0,048 per thousand contracts for the term structure delta and \$0,098 for the naive delta, i.e., the latter quantity is twice greater than the former. A relevant aspect to be perceived in the naive approach is that the hedging costs are biased (negatively from the perspective of the option seller). The bias is a main cause for such difference in performance between the strategies. It would perhaps be possible for the trader to evaluate beforehand the value of the bias. However, it would be difficult to introduce the information properly in the naive strategy in order to eliminate the bias and produce competitive hedging costs. Even considering the cancelation of the bias, a hedging cost measure of $0.048 per thousand contracts for the term structure delta and $0.054 for the naive delta would appear. Still, the term structure delta would perform 12.5% better that the naive procedure, in terms of the magnitude of the hedging error.

We also refer to the hedging cost measure offered in [1] - denoted exposure to risk index - which expresses the proportion of risk that is assumed by the trader to that absorbed by the discrete hedging strategy. It assigns a measure of quality of a given strategy per se and reads

$$v_{\Pi,\iota} = \frac{E[(e_\iota)^2]^{1/2}}{E[(e_\Pi)^2]^{1/2}} \tag{19}$$

In the above expression, Π denotes a strategy characterized by the fact that the trader hedges his / her position with the portfolio valued at the price of the option at time zero (C_0), totally invested in the money market account and do nothing more. Notice that the exposure to risk index of the strategy Π itself (i.e., with $\iota \equiv \Pi$) equals 100%.

In the above exercise, the exposure to risk index gives us the following picture: the term structure delta absorbs 66% of the risk affecting the trader's liability, leaving 34% for the trader to cope with (which is his / her exposure to risk). In turn, the naive delta only absorbs 28% of the risk leaving an exposure of 72% !

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5. Conclusions

We provide an analysis of dynamic hedging interest rate derivatives in the fixed income markets. We showed that the standard (or naive) approach of taking the numerical derivative of an option's price with respect to the underlying asset leads to a wrong delta value for higher order derivatives. In fact, this equals to assuming that we can fix the remainder points of the yield curve to evaluate the option's sensitivity, which is not true in the case of such derivatives. In the numerical examples, we use zero-coupon bond options - a representative of the class of higher order derivatives. Such examples unveil the fact that the difference between the naive (or standard) delta hedging procedure and the term structure one devised in the paper can breach the mark of 100% in favor of the latter, in terms of hedging error.

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