Klein-Gordon Equation and Wave Function in Robertson-Walker and Schwarzschild space-time

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Abstract: In the general relativity theory, we find Klein-Gordon wave functions in Robertson-Walker and Schwarzschild space-time. Specially, this article is that Klein-Gordon wave equations is treated by gauge fixing equations in Robertson-Walker space-time and Schwarzschild space-time.

Keywords: General relativity theory, Klein-Gordon wave equations; Klein-Gordon wave functions; Robertson-Walker space-time; Schwarzschild space-time

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1. INTRODUCTION

In the general relativity theory, our article’s aim is that we find Klein-Gordon wave equations and functions by gauge fixing equations in Robertson-Walker space-time and Schwarzschild space-time.

The gauge fixing equation in general relativity theory

\[ A^\mu_{\mu} = \frac{\partial A^\mu}{\partial \chi^\mu} + \Gamma^\mu_{\mu\rho} A^\rho \]

\[ \rightarrow \partial_\mu (A^\mu + g^{\mu\nu} \partial_\nu \Lambda) + \Gamma^\mu_{\mu\rho} (A^\rho + \partial^\rho \Lambda) \]

\[ = \partial_\mu (A^\mu + g^{\mu\nu} \partial_\nu \Lambda) + \Gamma^\mu_{\mu\rho} (A^\rho + g^{\rho\mu} \partial_\rho \Lambda) \]

(1)

2. KLEIN-GORDON WAVE EQUATION IN ROBERTSON-WALKER SPACE-TIME

Because the gauge fixing equation is the electro-magnetic wave equation, Klein-Gordon wave equation is in Robertson-Walker space-time.

The Robertson-Walker solution is

\[ d\tau^2 = dt^2 - \frac{1}{c^2} \Omega^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \]

(2)

In this time, 2-dimensional solution is

\[ d\Omega = 0 \]

\[ d\tau^2 = dt^2 - \frac{1}{c^2} \Omega^2(t) \frac{dr^2}{1 - kr^2} \]

(3)

The gauge fixing equation is in 2-dimensional solution[3]

\[ \partial_\mu (A^\mu + g^{\mu\nu} \partial_\nu \Lambda) + \Gamma^\mu_{\mu\rho} (A^\rho + g^{\rho\mu} \partial_\rho \Lambda) \]
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\[ \delta_{\mu} A^\mu + \Gamma^{10}_1 A^0 + \Gamma^1_{11} A^1 + \delta_{\mu} G^{\mu\nu} \partial_{\nu} \Lambda + g^{\mu\nu} \partial_{\mu} \partial_{\nu} \Lambda + \Gamma^{10}_1 G^{00} \frac{1}{c} \frac{\partial \Lambda}{\partial t} + \Gamma^1_{11} G^{11} \frac{\partial \Lambda}{\partial r} \]  

(4)

Hence, we can find Klein-Gordon wave equation in 2-dimentional Robertson-Walker space-time.

\[ \partial_{\mu} G^{\mu\nu} \partial_{\nu} \phi + g^{\mu\nu} \partial_{\mu} \partial_{\nu} \phi + \Gamma^{10}_1 g^{00} \frac{1}{c} \frac{\partial \phi}{\partial t} + \Gamma^1_{11} g^{11} \frac{\partial \phi}{\partial r} = \frac{m^2 c^4}{\hbar^2} \phi \]

(5)

In this time, we can think the shape of Klein-Gordon wave function from 2-dimentional Robertson-Walker space-time. In this case, light is

\[ d\tau^2 = dt^2 - \frac{1}{c^2} \Omega^2(t) \frac{dr^2}{1-kr^2} = 0 \]

(6)

Hence, matter wave function is in 2-dimentional Robertson-Walker space-time.

\[ \phi = A_0 \exp i \Phi, \quad A_0 \] is amplitude

\[ \Phi = \Omega \int \frac{dt}{\Omega(t)} - k_0 \int \frac{dr}{\sqrt{1-kr^2}}, \quad \omega_0 \] is angular frequency, \( k_0 = \left| k_0 \right| \) is wave number

i) \( k = 1, \phi = \Omega \int \frac{dt}{\Omega(t)} - k_0 \sin^{-1} r \)

ii) \( k = 0, \phi = \Omega \int \frac{dt}{\Omega(t)} - k_0 r \)

iii) \( k = -1, \phi = \Omega \int \frac{dt}{\Omega(t)} - k_0 \sinh^{-1} r \)

(7)

If the definition of energy and momentum is

\[ E = \frac{\hbar \omega_0}{\Omega(t)} \vec{P} = \frac{\hbar k_0}{\Omega(t)} \vec{r} \]

(8)

Energy-Momentum relation is in Robertson-Walker space-time,

\[ m^2 c^4 = E^2 - \frac{\Omega^2(t)}{1-kr^2} \vec{p}^2, E = m c^2 \frac{dt}{d\tau}, \vec{p} = m \frac{d\vec{r}}{d\tau} \]

(9)

Finally, angular frequency-wave number relation is in Robertson-Walker space-time,

\[ \hbar^2 \omega_0^2 - \frac{\hbar^2 k_0^2 c^2}{\Omega^2(t)} \frac{1}{1-kr^2} = m^2 c^4 \]

(10)

Hence, Klein-Gordon wave equation-Eq(5) is satisfied by matter wave function-Eq(7) in Robertson-Walker space-time.

3. KLEIN-GORDON WAVE EQUATION IN SCHWARZSCHILD SPACE-TIME

Because the gauge fixing equation is the electro-magnetic wave equation, Klein-Gordon wave
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The Schwarzschild solution is
\[ ds^2 = (1 - \frac{2GM}{rc^2})dt^2 - \frac{1}{c^2} \left[ \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\Omega^2 \right] \]  

(11)

In this time, 2-dimensional solution is
\[ d\tau^2 = (1 - \frac{2GM}{rc^2})dt^2 - \frac{1}{c^2} \frac{dr^2}{1 - \frac{2GM}{rc^2}} \]  

(12)

The gauge fixing equation is in 2-dimensional solution
\[ \partial_{\mu}(A^{\mu} + g^{\mu\nu} \partial_{\nu} \Lambda) + \Gamma^{\mu}_{\mu\nu}(A^{\nu} + g^{\mu\rho} \partial_{\rho} \Lambda) \]
\[ = \partial_{\mu}A^{\mu} + \Gamma_{01}^{01} A^1 + \Gamma_1^{11} A^1 + \partial_{\mu} g^{\mu\nu} \partial_{\nu} \Lambda + g^{\mu\nu} \partial_{\mu} \partial_{\nu} \Lambda + \Gamma_{01}^{01} g^{11} \frac{\partial \Lambda}{\partial r} + \Gamma_1^{11} g^{11} \frac{\partial \Lambda}{\partial r} \]
\[ = \partial_{\mu}A^{\mu} + \partial_{\mu} g^{\mu\nu} \partial_{\nu} \Lambda + g^{\mu\nu} \partial_{\mu} \partial_{\nu} \Lambda \]

\[ \Gamma_{01}^{01} = \frac{GM}{r^2 c^2} \frac{1}{1 - \frac{2GM}{rc^2}}, \quad \Gamma_1^{11} = -\frac{GM}{r^2 c^2} \frac{1}{1 - \frac{2GM}{rc^2}} \]  

(13)

Hence, we can find Klein-Gordon wave equation in 2-dimensional Schwarzschild space-time.
\[ \partial_{\mu} g^{\mu\nu} \partial_{\nu} \phi + g^{\mu\nu} \partial_{\mu} \partial_{\nu} \phi \]
\[ = \left( \frac{2GM}{r^2 c^2} - \frac{1}{c^2} \frac{1}{r^2 c^2} \frac{\partial^2}{\partial t^2} + \frac{1}{2GM} \frac{\partial^2}{\partial r^2} \right) \partial^2 \phi \]
\[ = \frac{m^2 c^4}{\hbar^2} \phi \]  

(14)

In this time, we can think the shape of Klein-Gordon wave function from 2-dimensional Schwarzschild space-time. In this case, light is
\[ d\tau^2 = (1 - \frac{2GM}{rc^2})dt^2 - \frac{1}{c^2} \frac{dr^2}{1 - \frac{2GM}{rc^2}} = 0 \]
\[ t = \frac{1}{c} \int \frac{dr}{1 - \frac{2GM}{rc^2}} = \frac{r}{c} + \frac{2GM}{c^3} \ln \left| r - \frac{2GM}{c^2} \right| \]  

(15)

Hence, Klein-Gordon wave function is in 2-dimensional Schwarzschild space-time-
\[ \phi = A_0 \exp i \Phi, A_0 \text{ is amplitude} \]
\[ \Phi = \omega_0 t - k_0 x - k_0 \frac{2GM}{c^2} \ln \left| x - \frac{2GM}{c^2} \right| \]
\[ \omega_0 \text{ is angular frequency}, \quad k_0 = \left| k_0 \right| \text{ is wave number} \]  

(16)
If the definition of energy and momentum is

\[ E = \frac{\hbar \omega_0}{(\xi - \frac{2GM}{rc^2})} \vec{p} = \hbar \vec{k}_0 \left( \xi - \frac{2GM}{rc^2} \right) \]  

(17)

Energy-Momentum relation is in Schwarzschild space-time,

\[ m^2 c^4 = \left( \xi - \frac{2GM}{rc^2} \right) E^2 - \frac{\vec{p}^2 c^2}{\left( \xi - \frac{2GM}{rc^2} \right)} \]

\[ \vec{E} = m c^2 \frac{dt}{d\tau} \vec{r} = m \frac{d\vec{r}}{d\tau} \]  

(18)

Finally, angular frequency-wave number relation is in Schwarzschild space-time,

\[ \frac{\hbar^2 \omega_0^2}{(\xi - \frac{2GM}{rc^2})} - \hbar^2 \vec{k}_0^2 \left( \xi - \frac{2GM}{rc^2} \right) = m^2 c^4 \]  

(19)

Hence, Klein-Gordon wave equation-Eq(14) is satisfied by matter wave function-Eq(16) in Schwarzschild space-time.

4. CONCLUSION

We find Klein-Gordon wave equation and function in Robertson-Walker space-time. We find Klein-Gordon wave equation and function in Schwarzschild space-time.

REFERENCES


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